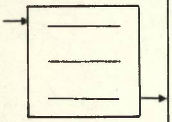
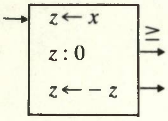
Elementary Functions:

an algorithmic treatment

Kenneth E. Iverson

Summary of Notation

Function	Notation	Definition or Example	Page Refs.	Computer Notation
Specification	$z \leftarrow x$	$z \leftarrow 3$ assigns the value 3 to z	6	$Z \leftarrow X$
Arithmetic	$+ - \times \div$		6	$+ - \times \div$
Branch	$\|x : y\|\xrightarrow{\mathscr{R}}$	Arrow is followed if $x \mathscr{R} y$ is true	11	$\rightarrow (X \mathscr{R} Y)/ S$
Relations $\mathscr{R}$	$< \leq = \geq > \neq$	in branches and relational functions	6	$< \leq = \geq > \neq$
Component of $\boldsymbol{x}$	$\boldsymbol{x}_i$	ith component of $\boldsymbol{x}$	18	$X[I]$
Dimension of $\boldsymbol{x}$	$\rho \boldsymbol{x}$	$\rho\ (3, 4, 5, 6) \equiv 4$	18	ρX
Catenation	$\boldsymbol{x}, \boldsymbol{y}$	$\boldsymbol{x}, \boldsymbol{y} \equiv \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{\rho \boldsymbol{x}}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{\rho \boldsymbol{y}}$	18	X, Y
Definition of function F	$z \leftarrow F\, x$ 	$z \leftarrow \|x$ 	39	$\nabla Z \leftarrow F\ X$ [1] ——— [2] ——— [3] ——— [4] ∇
Maximum	$x \lceil y$	$4 \lceil 2 \equiv 4$	40	$X \lceil Y$
Minimum	$x \lfloor y$	$4 \lfloor 2 \equiv 2$	43	$X \lfloor Y$
Residue	$m\|n$	$3\|7 \equiv 1;\ 3\|{}^{-}7 \equiv 2;\ 3\|6 \equiv 0$	43	$X \| Y$
Absolute value	$\|x$	$\|3.14 \equiv 3.14;\ \|{}^{-}3.14 \equiv 3.14$	43	$\|X$
Negation	$-x$	$-x \equiv 0 - x$	43	$-X$
Exponentiation	$x * n$	$x * 0 \equiv 1;\ x * n \equiv x \times x * n - 1$	45	$X * N$
Factorial	$!\,n$	$!\,0 \equiv 1;\ !\,n \equiv n \times !\,n - 1$	45	$!\,N$
Relation	$x \mathscr{R} y$	$(3 \leq 3) \equiv 1;\ (3 < 3) \equiv 0$	47	$X \mathscr{R} Y$
Compression	$\boldsymbol{u} / \boldsymbol{x}$	$(1, 0, 1, 0, 1)/ \boldsymbol{x} \equiv (\boldsymbol{x}_1, \boldsymbol{x}_3, \boldsymbol{x}_5)$	48	U / X
Reversal	$\Phi\, \boldsymbol{x}$	$\Phi\, 1, 2, 3, 4 \equiv 4, 3, 2, 1$	48	$\Phi\, X$
Integer vector	$\iota\, n$	$\iota\, 4 \equiv 1, 2, 3, 4$	48	$\iota\, N$
Reduction	$F/ \boldsymbol{x}$	$F/ \boldsymbol{x} \equiv \boldsymbol{x}_1\, F\, \boldsymbol{x}_2\, F\, \boldsymbol{x}_3 \ldots F\, \boldsymbol{x}_{\rho \boldsymbol{x}}$	22	F/ X
Row i of matrix	$\boldsymbol{M}^i$	$\boldsymbol{M}^2 \equiv 4, 5, 6$	76	$M[I;]$
Column i of matrix	$\boldsymbol{M}_i$	$\boldsymbol{M}_2 \equiv 2, 5, 8, 11$	76	$M[;I]$
Element of matrix	$\boldsymbol{M}^i_j$	$\boldsymbol{M}^2_3 \equiv 6$ $\quad \left.\right\} \boldsymbol{M} \equiv \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{matrix}$	76	$M[I;J]$
Restructuring	$\boldsymbol{d}\, \rho\, \boldsymbol{x}$	$(4, 3)\, \rho\, \iota\, 12 \equiv \boldsymbol{M}$ $12\, \rho\, \boldsymbol{M} \equiv \iota\, 12$	79	$D\, \rho\, X$
Polynomial	$\boldsymbol{c}\, \Pi\, x$	$\boldsymbol{c}_1 + (\boldsymbol{c}_2 \times x) + (\boldsymbol{c}_3 \times x^2) + \ldots$	62	
Natural exponential	$* x$	$\left(1, 1, \frac{1}{!2}, \frac{1}{!3}, \frac{1}{!4}, \ldots\right) \Pi\, x$	187	$* X$
Hyperbolic cosine	$A\, x$	$\left(1, 0, \frac{1}{!2}, 0, \frac{1}{!4}, \ldots\right) \Pi\, x$	180	
Hyperbolic sine	$B\, x$	$\left(0, 1, 0, \frac{1}{!3}, 0, \ldots\right) \Pi\, x$	180	
Cosine	$C\, x$	$\left(1, 0, \frac{{}^{-}1}{!2}, 0, \frac{1}{!4}, \ldots\right) \Pi\, x$	133	
Sine	$S\, x$	$\left(0, 1, 0, \frac{{}^{-}1}{!3}, 0, \frac{1}{!5}, \ldots\right) \Pi\, x$	133	
Tangent	$T\, x$	$(S\, x) \div C\, x$	145	
Hyperbolic tangent	$U\, x$	$(B\, x) \div A\, x$	180	
Base of natural logarithm	e	$2.71828 \ldots \equiv 1 + \frac{1}{!2} + \frac{1}{!3} + \ldots$	187	
Circular constant	π	$3.14159 \ldots$	15	

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Preface

The present text is intended for a one-semester precalculus course at the freshman level. The main topics are those to be expected in an introductory course in elementary functions—polynomials, circular functions, and the logarithmic and exponential functions.

The major distinguishing characteristic of this treatment is the systematic use of formal algorithms or *programs* in the definition of functions. Programs are expressed in a simple programming language based on a small set of simple primitives: +, −, ×, ÷, specification, branch, selection of a component of a vector, and formal composition of definitions.

The following characteristics of the treatment are also of interest:

1. Vectors are introduced early as an effective device for treating a family of variables and are used throughout the text. For example, the polynomial is treated in terms of its vector of coefficients, yielding simple explicit procedures for addition and multiplication of polynomials, synthetic division, and the generation of Pascal's triangle.
2. The slope of a function is also defined early and is developed as a powerful tool for the study of functions. Use of the slope function yields, in particular, simple derivations of the polynomial expansions of all the functions treated.
3. Notation is introduced for the reciprocal of a function and the inverse of a function and is used to clarify the relations among the functions treated.
4. The mathematical derivations are kept simple. Many are novel—for example, the derivation of the slope of the reciprocal function in Chapter 7.

5. Each step of the development is carefully motivated; spurious rigor (that is, the making of careful distinctions which are never actually needed or used at the particular level of presentation) is avoided.

Programming is learned as a by-product of the constant use of algorithms. The use of a computer (treated in Chapter 9) is helpful but is neither necessary nor central to the development. The exercises are numerous (over 250) and span a wide range of difficulty. A complete booklet of solutions is available from the publisher.

This book grew from my own work in programming languages, and from my conviction that the discipline naturally imposed by the formalism required in programming would prove a boon in the exposition of mathematics from a very early level onward. It was developed in a one-semester course for seniors at the Fox Lane High School.

I am indebted to a number of the faculty of the Fox Lane High School—to Mr. Robert Wallace for initial discussions of course content, to Mr. George Kasunick, Dr. Norman Michaels, and Dr. Neil Atkins for their support and cooperation, and to Messrs. Harold Barrett and James Lott for many helpful discussions. To Dr. Herbert Hellerman of the IBM Systems Research Institute, I am indebted for the use of his PAT system on the 1620 computer and for the many evenings he gave to coaching students in its use during the first years of the course. For the programming system used in the third year of the course, I am obliged to Mr. Lawrence M. Breed of IBM and Mr. Philip S. Abrams of Stanford University. I am particularly indebted to Mr. Adin D. Falkoff and to Mr. Breed, my colleagues at the Thomas J. Watson Research Center, for many helpful suggestions and discussions. Thanks are also due to the International Business Machines Corporation and, in particular, to Dr. G. L. Tucker, for free time to teach the course. I am also pleased to acknowledge the editorial assistance of Messrs. Peter Saecker and Coley Mills of Science Research Associates. I am indebted to my wife Jean for typing and retyping the manuscript and to my son Eric for preparing solutions for the exercises.

Mount Kisco, New York KENNETH E. IVERSON
March 1, 1966

Contents

Chapter One

Introduction

In order to give an overall view of the matters to be treated in this course it is necessary to begin with a simple working definition of the mathematical term *function.*

The weight of a quantity of water is related to its volume in the manner indicated in Table 1.1; for any volume shown in the table, the weight can be determined, and the weight of the water is said to be a *function* of its volume. The table could, of course, be enlarged to include fractional volumes so that the weights of any selected set of volumes would be specified. In general, if the value of any quantity w (in this case, weight) is determined by the value of some other quantity v (in this case, volume), then w is said to be a *function* of v. The variable v is said to be the *argument* of the function.

Volume (cubic inches)	Weight (ounces)
1	0.575
2	1.150
3	1.725
4	2.300
5	2.875
6	3.450
.	.
.	.
.	.

Table 1.1 Weight of water as a function of volume

A function can be specified in several ways. The function of Table 1.1 can, for example, be specified either by the graph in Figure 1.2 or by the equation

$$w = 0.575 \times v \tag{1.1}$$

Determining the value of w that corresponds to some chosen value of the argument v is called *evaluating* the function. In the present example the function can be evaluated in three ways: by looking up the value in Table 1.1, by reading off the value of the corresponding point on the graph of Figure 1.2, or by computing the product $0.575 \times v$ indicated in Equation 1.1.

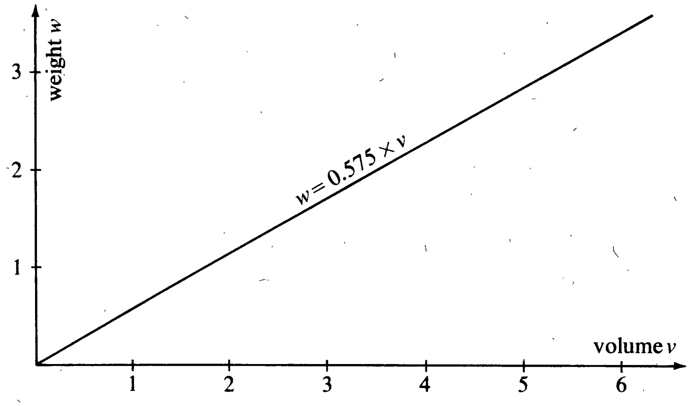


Figure 1.2 Graphical representation of the function of Table 1.1

Anyone wishing to check the specified relation between weight and volume of water could perform experiments with an accurate spring scale. If he performs the experiments near the North Pole, however, he will find the slightly different relation shown in Table 1.3. Since this relation is different, must one conclude that the weight is not really a function of the volume? No, one may suppose that the weight depends not only on the volume but also on some second factor. In this example it happens that the second factor is the strength of the earth's gravitation, which is greatest at the poles and decreases toward the equator. Thus the weight of water can be considered a function of *two* arguments, the volume v and the gravitation g:

$$w = .0179 \times g \times v$$

This can be shown to agree with both Tables 1.1 and 1.3 by using the appropriate values of g, which are 32.08 and 32.25 at the equator and the pole respectively.

Volume (cubic inches)	Weight (ounces)
1	0.578
2	1.156
3	1.734
4	2.312
5	2.890
6	3.468
.	.
.	.
.	.

Table 1.3 Weight of water as a function of volume (at North Pole)

A function can have one, two, three, four, or even more arguments. For example, the volume v of a rectangular box is a function of three arguments: the length l, the breadth b, and the height h. This function can be expressed as follows:

$$v = l \times b \times h$$

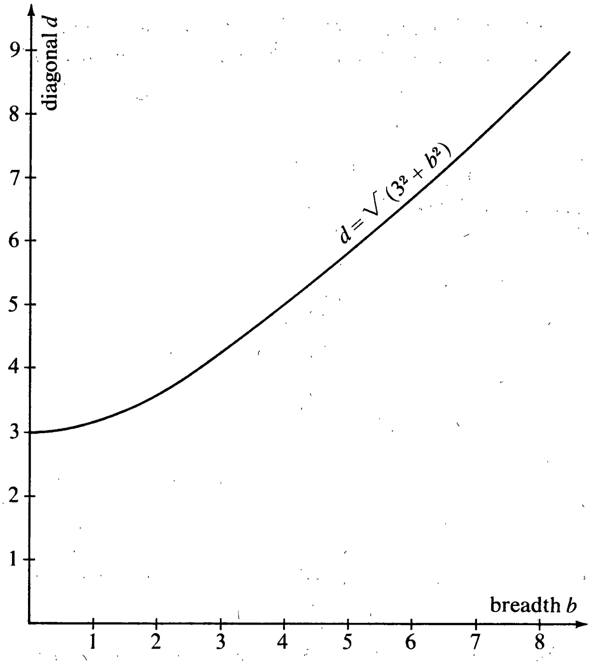
Furthermore, the weight of water filling such a box, which was treated as a function of the two arguments g and v, can also be considered a function of four arguments g, l, b, and h:

$$w = .0179 \times g \times l \times b \times h$$

Each of the functions mentioned thus far has been expressed as a simple *product* of its arguments. This is, of course, not possible in general. For example, the length d of a diagonal of a rectangle with sides of length 3 and length b is the following function of b:

$$d = \surd\ (3^2 + b^2)$$

The graph of this function (Figure 1.4) is a curve, unlike the straight-line graph of Figure 1.2.



b	d
0	3.00
1	3.16
2	3.61
3	4.24
4	5.00
5	5.83
6	6.71
7	7.61
8	8.54

Figure 1.4 Three representations of a function

The simple notion of a *function* proves extremely useful, not only in such obvious areas as science and engineering (the weight a suspension bridge can support, for example, is a function of many arguments, such as the spacing of the towers, the size of the cables, and the quality of the steel), but also, though less obviously, in the humanities and virtually every field of study. Even the familiar operations of arithmetic (addition, subtraction, multiplication, and division) are functions.

Although there is a vast number of functions of practical interest, most of them can be expressed, exactly or approximately, in terms of a rather small number of *elementary* functions. Since the elementary functions are the simple building blocks from which more complex functions are constructed, their study is an important part of mathematics. This course will concern the simplest and most important elementary functions: *polynomial* functions, *circular* (trigonometric) functions, *logarithmic* functions, *exponential* functions, and *hyperbolic* functions.

The first step will be to develop a notation for describing or *defining* functions. As illustrated in the preceding examples, algebraic expressions are useful in defining some functions. The notation to be developed is a simple extension of algebraic notation and permits the use of a sequence of expressions that are to be evaluated in a specified order.

The sequence of expressions that defines a function actually provides a recipe, or *algorithm*, for calculating the value of the function for any given values of the arguments. Since one is frequently concerned with evaluating a function, it is advantageous to have the function defined in such a form. Furthermore, modern computers can be made to obey such a recipe automatically.

Chapter 9 deals with the use of the computer to evaluate functions. If a computer is not available, this chapter may well be omitted. Chapter 9 depends on both Chapters 2 and 3 but may be studied concurrently with them.

Chapter Two

Programming Notation

An adequate notation for the treatment of functions can be developed by a few simple extensions and modifications of familiar algebraic notation. The resulting notation is summarized in Appendix D on page 226. Although intended primarily for later reference, this summary should also prove helpful in reading this chapter.

Specification

The basic relations between numbers are represented by the familiar symbols $<, \leq, =, \geq, >$, and $\neq$. For example, $x \leq 14$ means that the value of x does not exceed 14, and $y = x + 3$ means that y has the same value as $x + 3$.

The symbol = is also used in a related but quite different sense that must be distinguished from the *relation* of equality. To illustrate the second use of =, consider the following prescription for calculating the area a of a rectangle of length l and width w:

$$a = l \times w$$

If the symbol = represents a relation, then the expression simply means that the area a is equal to the expression on the right. This is, of course, true; but more is implied, namely, that the variable a is to be *made* equal to the value of the expression on the right. The distinction is the same as that between the indicative mood in English ("The door is closed") and the imperative ("Close the door"). The imperative orders an action to be performed, at the conclusion of which the corresponding indicative statement ("The door is closed") is, of course, true.

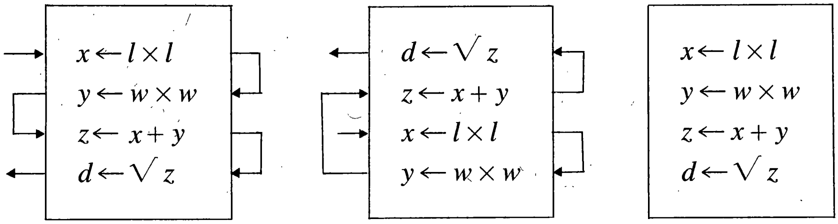
The notation

$$a \leftarrow l \times w$$

will be adopted to denote that a is to be *made* equal to $l \times w$. The arrow denotes *specification* and implies that the value of the expression on the right *specifies* the value of the variable on the left. The entire expression is read either as "a is specified by $l \times w$" or as "$l \times w$ specifies a"; it has the force of an imperative sentence and will be called a *statement*. Performance of the indicated specification is called *execution* of the statement.

Programs

A list of statements, together with a set of sequence arrows indicating the order in which they are to be executed, is called a *program*. For example, Program 2.1 (a) prescribes a calculation of the length of the diagonal d of the rectangle in Figure 2.2. The meaning of a sequence arrow is obvious: an *entrance arrow* (as at the left of the first statement of Program 2.1 (a)) indicates the first statement to be executed; an arrow joins each statement to its successor (that is, the next



(a) (b) (c)

Program 2.1 Three equivalent programs

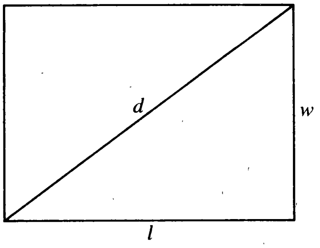


Figure 2.2 The diagonal of a rectangle

to be executed); and an *exit arrow* (as at the left of the last statement) indicates the end of the program. It is therefore clear that Program 2.1 (b) is equivalent to 2.1 (a). To simplify matters in common cases, the entrance arrow to the first statement in a list can be omitted, any sequence arrow from a statement to the one immediately following it can be omitted, and the exit arrow from the last statement can be omitted. With these rules, Program 2.1 (c) is also seen to be equivalent to 2.1 (a).

The validity of these programs can be tested by carrying out the steps indicated by the successive statements for some assumed values of l (length) and w (width). For example, if $l = 4$ and $w = 3$, the record would appear as follows:

l	4
w	3
x	16
y	9
z	25
d	5

The making of such a complete step-by-step record is called *executing* the program, and the record itself is called an *execution* of the program.

The initial values of the arguments l and w can be indicated either informally, as was done in the execution above, or formally by prefacing the program with the statements $l \leftarrow 4$ and $w \leftarrow 3$ as in the program below.

```
→ $l \leftarrow 4$
  $w \leftarrow 3$
  $x \leftarrow l \times l$
  $y \leftarrow w \times w$
  $z \leftarrow x + y$
  $d \leftarrow \surd z$ →
```

The sequence of execution can also be indicated in the familiar manner by parentheses; the calculation of d could, for example, be written as a one-statement program as follows:

$$d \leftarrow \surd((l \times l) + (w \times w))$$

Another familiar rule states that multiplication is performed before addition, so that the foregoing could have been written with fewer parentheses as follows:

$$d \leftarrow \sqrt{\ } (l \times l + w \times w)$$

Such a rule becomes less useful when many further functions are introduced. Consequently this rule will *not* be used. However, in order to make every statement unambiguous, the following rule *will* be adopted: The functions in a statement will be evaluated in order from right to left, subject, of course, to the order indicated by parentheses. For example, the statement

$$t \leftarrow (s - q) + r \times s - p$$

is equivalent to the fully parenthesized statement

$$t \leftarrow (s - q) + (r \times (s - p))$$

Note that the rule is to evaluate the functions from right to left, not to *read* the entire expression from right to left; thus the last term of the preceding expression remains $(s - p)$, *not* $(p - s)$.

This change in convention, although awkward at first, can be assimilated quickly and accurately by first parenthesizing all statements completely and then gradually omitting parentheses as the right-to-left rule becomes more familiar.†

Alternative symbols for the same function will be avoided; thus $x \cdot y$ and xy will *not* be used for $x \times y$, and x/y will *not* be used formally for $x \div y$. Moreover, since uppercase (capital) letters will be used to denote *functions*, variables will be denoted by lowercase letters only. To make the distinction clear, students should perhaps use script letters for lowercase; this also avoids the common confusion between x and $\times$.

Any variable can be specified and respecified any number of times in the course of executing a program, and the value of the variable is that produced by the last specification performed in the program. For example, the "intermediate" variables x, y, and z, used in the original program for calculating d, can be eliminated by using the following equivalent program:

---

†This departure from established mathematical convention has not been adopted lightly. The reasons for it are discussed fully in Appendix A.

```
$d \leftarrow l \times l$
$d \leftarrow d + w \times w$
$d \leftarrow \sqrt{\ } d$
```

For the case $l = 4$, $w = 3$ the execution now appears as

l	4
w	3
d	$\cancel{16}$ $\cancel{25}$ 5

When a variable (such as d) appears on both sides of a specification arrow, the value given to it on the right is its old value (previously assigned), and its value on the left is the new value (obtained by evaluating the entire expression on the right).

Finally, it is important to emphasize that every properly written program is a *complete and explicit* statement of a calculation; after the initial specification of the values of certain of the variables (the *arguments*) it must be possible to execute the entire program mechanically, *using only the few rules and functions detailed above, without recourse to any other knowledge.* If any undefined function, symbol, or unspecified variable occurs in the program, execution stops at that point.

Writing a complete execution of a program is a powerful method of gaining a full understanding of the behavior and purpose of the program. For every new program encountered in the text the reader should carry out one or more executions.

Interpretation Tables

In writing or using a program it is usually helpful to have a list of the variables used, together with the significance attached to each variable. Such a list is called an *interpretation table*; in the case of Program 2.1 (a) it would appear as follows:

Significance	Variables	Execution
Length of rectangle	l	
Width of rectangle	w	
Square of length of rectangle	x	
Square of width of rectangle	y	
Square of diagonal of rectangle	z	
Length of diagonal of rectangle	d	

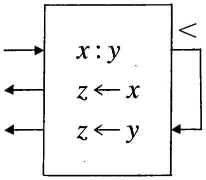
The interpretation table is helpful in interpreting the purpose and behavior of a program. The information in the left-hand column is strictly auxiliary, however, and has no bearing on the execution of a program.

(Do Exercises 2.1–2.8.)

Variable Sequence: Branching

The simple notation already introduced can describe a wide variety of calculations (as indicated by the programs of Exercises 2.1 through 2.8), but it cannot cope with the simple problem of determining the maximum of two arguments. However, a simple extension of the notation not only remedies this particular deficiency but greatly increases its power.

The trick is to introduce a new type of statement, called a *comparison* or *conditional branch* statement, which can change the sequence in which the statements of the program are executed. The conditional branch will first be illustrated by using it in a program for determining z as the maximum of two arguments x and y:



The symbol < used to label the sequence arrow on the right indicates that the arrow is followed if and only if a true statement is obtained when the colon is replaced with the < in the accompanying comparison statement. Thus if $x<y$, the third statement is executed next; if $x \geq y$, the second statement is executed. For example, if $x=3$ and $y=5$, then $x<y$ and statement three alone is executed to yield $z=5$; if $x=3$ and $y=2$, then $x \geq y$ and statement two alone is executed to yield $z=3$. The executions for several cases appear as follows:

x	3	x	3	x	3	x	$^{-}3$
y	5	y	2	y	3	y	$^{-}5$
z	5	z	3	z	3	z	$^{-}3$

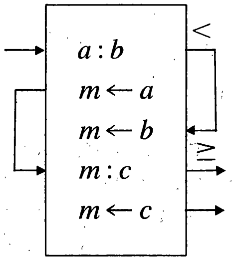
A comparison statement does not respecify the value of any variable; it merely affects the sequence in which the program statements are executed.

More than one labeled sequence arrow may accompany a single comparison statement; however, in order that the sequence of execution of the program (and hence its results) be unambiguous, it is necessary that these relations be mutually exclusive. For example, any one of the pairs $<, >$; $<, \geq$; and $=, \neq$ may be used to label two branches from a single comparison statement, but the pair $>, \geq$ may not. The arrow in the unlabeled branch introduced earlier is always followed; this can be considered an *unconditional branch.*

As a further example of the use of the branch, consider the following interpretation table and execution:

Significance	Variables	Execution
Weight of Andy	a	130
Weight of Bob	b	150
Weight of Charles	c	125
Maximum of the weights of the three boys	m	150

and the corresponding program, Program 2.3.



(Do Exercises 2.9–2.12.)

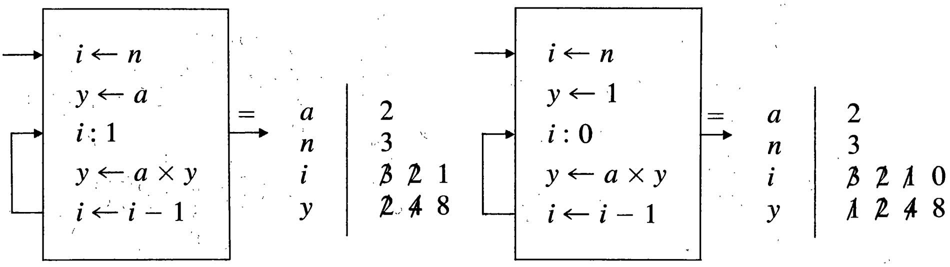
Program 2.3 The maximum of three arguments

The branch statement will prove useful in a surprising number of ways, but before examining them it is desirable to collect in one table the notation introduced thus far; until further notation is introduced, all statements used in programs will be limited to the types shown in Table 2.4. In particular, it should be noted that the exponential notation (x^2, x^3, and so on) is not included. This notation will be used informally in the text, but must not be used in program statements.

Specification	$z \leftarrow x$
Branch	$\vert\, x : y \;\mathscr{R}$
Relations $\mathscr{R}$	$< \leq = \geq > \neq$
Multiplication	$x \times y$
Division	$x \div y$
Addition	$x + y$
Subtraction	$x - y$
Negation	$-y$
Square root†	$\sqrt{x}$

Table 2.4 Summary of notation

How then is a function such as "the nth power of a" (that is, a^n) to be represented? Program 2.5 (a) shows one possible method employing the conditional branch. Execution for the case $a = 2$, $n = 3$ yields $y = 2^3 = 8$, as shown to the right of the program.



(a) (b)

Program 2.5 The nth power of a

Executing Program 2.5 (a) for one or two further cases (say $a = 3$, $n = 3$ and $a = 3$, $n = 1$) will clarify its behavior. The heart of the process is statement 4, which is repeated or *iterated* the required number of times as determined by the value of n. Such a process is called *itera-*

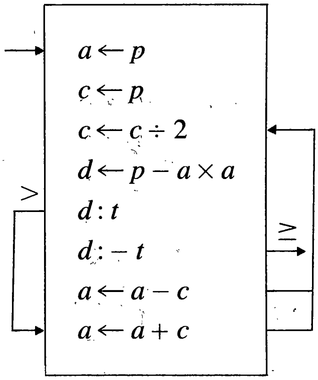
---

†The symbol $\sqrt{}$ for square root will not be used on the computer (see Chapter 9) and will shortly be supplanted by a more general expression.

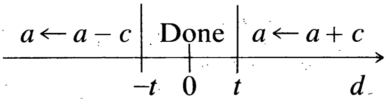
tive, and the portion of the program that is iterated (statements 3 through 5) is called a *loop*.

Although it works perfectly for positive integral values of n, Program 2.5 (a) is defective; it does not give the correct result (and in fact never terminates) for the case $n = 0$. Program 2.5 (b) corrects this defect but is still limited to nonnegative integral values of n.

An example of another important use of iteration is furnished by the related problem of determining a as a function of the argument p, so that a is the nth root of p (that is, $a = \sqrt[n]{p}$ or $p^{1 \div n}$). Program 2.6 shows one solution for the case $n = 2$, that is, for square root. The basic idea is obvious from the program: the variable a is an approximation to the required solution, which is improved at each iteration by either adding or subtracting a chosen correction c. If the difference $d = p - a^2$ is positive, the correction is added; otherwise it is subtracted. The solution will not be exact (for example, if $p = 8$, the exact solution $a = 2.828 \ldots$ is an endless decimal) and it is therefore necessary to terminate the process when the difference d is less than a specified *tolerance* t. The accompanying execution and sketch should make the behavior clear.



t | 0.2
p | 8
c | ~~8~~ ~~4~~ ~~2~~ ~~1~~ ~~.5~~ ~~.25~~ ~~.125~~ ~~.0625~~ .03125
a | ~~8~~ ~~4~~ ~~2~~ ~~3~~ ~~2.5~~ ~~2.75~~ ~~2.875~~ 2.8025
d | ~~⁻56~~ ~~⁻8~~ ~~4~~ ~~⁻1~~ ~~1.75~~ ~~.4375~~ ~~⁻.265625~~ .14599375



Program 2.6 The square root of p

Program 2.6 will correctly obtain the square root of p for all values of p greater than 1 but will not work for small values such as 0.1. This problem is examined and solved in Exercise 2.14, as is the problem of extending the program to obtain the nth root of p for any integer n.

In executing any program involving a tolerance in the manner of Program 2.6, it is clearly unnecessary to carry the calculations to a degree of accuracy much greater than the tolerance. The computer operation described in Chapter 9 carries calculations to an accuracy of eight decimal places. In exercises, an accuracy of four decimal digits should suffice.

(Do Exercises 2.13–2.15.)

A more interesting example of the use of successive approximations is furnished by the following method for computing the value of π. From Figure 2.7 it is clear that the perimeter $p = 6 \times a$ of the inscribed hexagon is a rough approximation to the circumference of the circle of radius 1 (and hence to the value of $2 \times \pi$). Furthermore, the perimeter $p' = 12 \times a'$ of the inscribed dodecagon is a better approximation, and $p'' = 24 \times a''$ is even better, and so on.

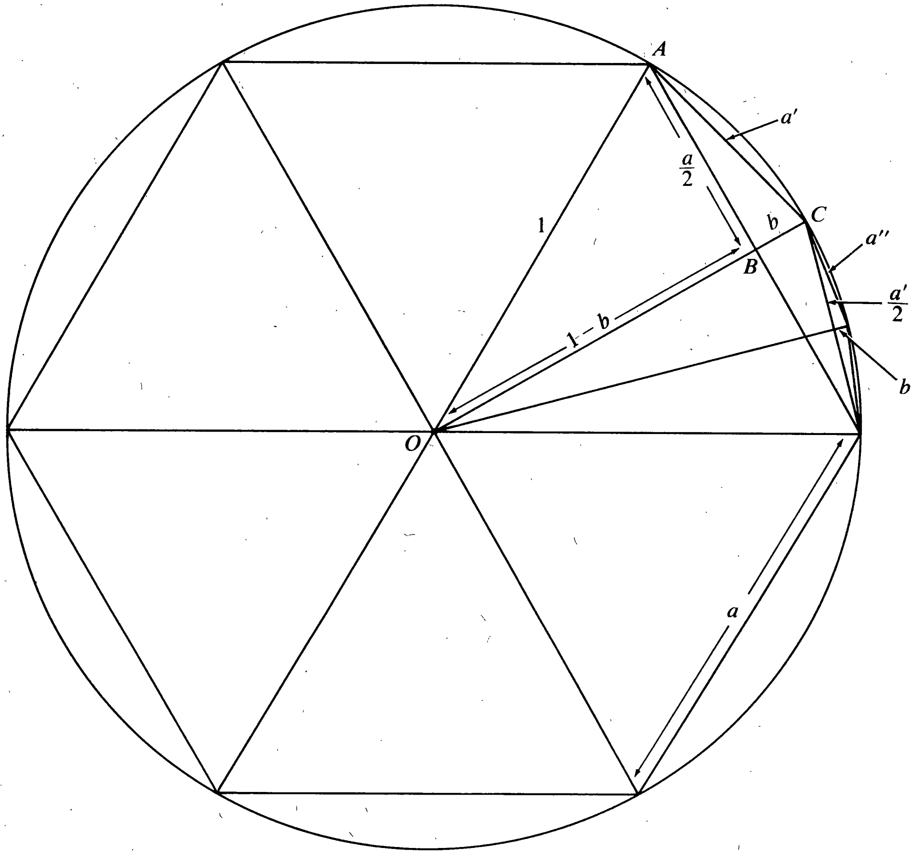


Figure 2.7 Approximating $2 \times \pi$ by inscribed polygons

From the right triangle *ABO* it is clear that

$$1 - b = \surd\, 1 - a^2 \div 4$$

Consequently

$$b^2 = (1 - \surd\, 1 - a^2 \div 4)^2 = (1 - a^2 \div 4) + 1 - 2 \times \surd\, 1 - a^2 \div 4$$

and

$$\begin{aligned} b^2 + a^2 \div 4 &= 2 - 2 \times \surd\, 1 - a^2 \div 4 \\ &= 2 - \surd\, 4 - a^2 \end{aligned}$$

From triangle *ABC*, $a' = \surd\, b^2 + a^2 \div 4$. Hence,

$$a' = \surd\, 2 - \surd\, 4 - a^2$$

But a'' is obtained from a' in the same way that a' was obtained from a. Hence

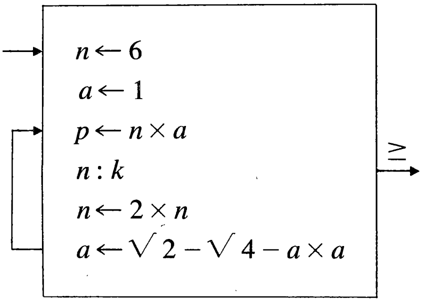
$$\begin{aligned} a'' &= \surd\, 2 - \surd\, 4 - (a')^2 \\ a''' &= \surd\, 2 - \surd\, 4 - (a'')^2 \end{aligned}$$

and so on.

In a program it is convenient to use the same variable a to represent all the successive approximations. Thus the expressions for a', a'', a''', and so forth, can all be replaced by

$$a \leftarrow \surd\, 2 - \surd\, 4 - a^2$$

Program 2.8 describes the process for approximation by a k-sided polygon.



```
n ← 6
a ← 1
p ← n × a
n : k
n ← 2 × n
a ← √ 2 − √ 4 − a × a
```

(Do Exercises 2.16 – 2.21.)

Program 2.8 An approximation to $2 \times \pi$

Notation for Numbers

All nonnegative numbers used in programs will be denoted by decimal notation in the usual way except that commas will not be permitted between groups of digits. Thus 1231 and 12.31 and .1231 and 0.1231 are permitted, but 1,231 is not permitted as alternative notation for 1231.

A rational number such as *two-thirds* will be denoted informally in exposition by $\frac{2}{3}$, but in a program it must be denoted formally by division (that is, $2 \div 3$) or by an approximate decimal value such as 0.667.

The treatment of negative numbers can be clarified by adopting for them a notation which is distinct from the notation for negating a positive number†. This notation (which will now be adopted) consists of a *negative sign* ($^{-}$) preceding the number, thus: $^{-}1$, $^{-}2$, $^{-}3$, $^{-}3.1416$, $^{-}0.5$, $^{-}.5$, and so forth. The raised negative symbol ($^{-}$) does not denote a function as does the *minus sign*, but is rather an integral part of the representation of the number just as the decimal point is. Therefore it is as meaningless to write ^{-}x or $^{-}{}^{-}3$ or $^{-}(3)$ or $^{-}-3$, as it is to write $.x$ or $..3$ or $.(3)$ or $.-3$. It is of course meaningful to write $-{}^{-}3$ (which is equal to 3), as it is to write $-.3$. The negative sign is meaningful only if it immediately precedes a digit or a decimal point. The number $^{-}3$ will be read as "negative three," whereas -3 will be read as "minus three."

Vectors

The example of Program 2.3 (for finding the maximum of three boys' weights a, b, c) can be extended without difficulty to four boys by employing arguments a, b, c, d, or to five by using a, b, c, d, e, and so on for any number of arguments. If, however, the number of arguments is itself a variable (for example, the number of boys present in class tomorrow), a new difficulty arises. This can be surmounted by adopting a single name for such a *family* of variables and by identifying each member of the family by a numerical subscript or *index*. Thus, if $\boldsymbol{x}$ is the name of a family of four members, the successive members are $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_4$.

Such a family of variables will be called a *vector* and each mem-

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†See M. Beberman and H. E. Vaughan, *High School Mathematics Course 1* (Heath, 1964), or D. E. Morris and H. D. Topfer, *Advancing in Mathematics, J2* (Science Research Associates, 1964).

ber will be called a *component* of the vector. For example, if $\boldsymbol{w}$ is a vector of the weights of three boys, the first of whom weighs 130 pounds, the second 150, and the third 125, then $\boldsymbol{w}_1 = 130$, and $\boldsymbol{w}_2 = 150$, and $\boldsymbol{w}_3 = 125$. The number of components in a vector $\boldsymbol{x}$ is called the *dimension* of $\boldsymbol{x}$ and is denoted by $\rho\, \boldsymbol{x}$.† The dimension of a vector is clearly a function of that vector; in the current example, $\rho\, \boldsymbol{w}$ is 3.

A single variable will now be called a *scalar* to distinguish it from a *vector.* A vector will be denoted by a boldface lowercase italic letter (for example, $\boldsymbol{w}$) to distinguish it from a scalar, which will be denoted by a lightface lowercase italic as before. A component of a vector $\boldsymbol{w}$ is itself a scalar but it will also be shown in boldface (for example, $\boldsymbol{w}_3$) to indicate that it is a member of the family. In handwriting one may distinguish a vector by an underscore: $\underline{x}$.

A vector can be formed by the catenation (chaining together) of scalars. The catenation function is denoted by a comma. Thus the statement

$$\boldsymbol{f} \leftarrow 3\,,5$$

specifies $\boldsymbol{f}$ as a vector of dimension 2 such that $\boldsymbol{f}_1 = 3$ and $\boldsymbol{f}_2 = 5$ and, as stated, $\rho\, \boldsymbol{f} = 2$. Catenation is defined on vectors as well as on scalars, so that if

$$\boldsymbol{g} \leftarrow 7\,,9 \text{ and } \boldsymbol{h} \leftarrow \boldsymbol{f}, \boldsymbol{g}$$

then

$$\rho\, \boldsymbol{h} = 4$$

and

$$\boldsymbol{h}_1 = 3 \qquad \boldsymbol{h}_2 = 5 \qquad \boldsymbol{h}_3 = 7 \qquad \boldsymbol{h}_4 = 9$$

Successive catenations are executed in the established order, that is, from right to left. Hence the statement

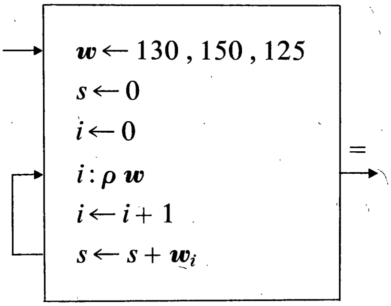
$$\boldsymbol{k} \leftarrow 3\,,5\,,7\,,9$$

yields a vector $\boldsymbol{k}$ which is identical with the vector $\boldsymbol{h}$ of the preceding paragraph. The common practice of *requiring* parentheses enclosing the set of components of a vector will not be followed. Extra parentheses can of course be placed around any expression—including the catenation denoting a vector.

---

†The letter ρ is the Greek letter corresponding to the Roman *r*. It is spelled *rho* and pronounced *roe*.

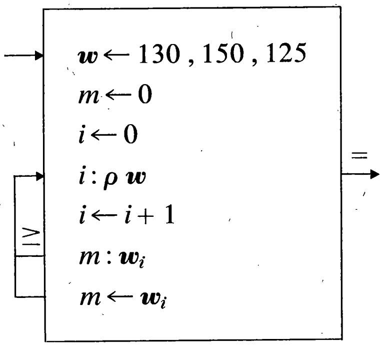
To illustrate the use of a vector, consider the following program, which determines the sum s of all components of the vector $\boldsymbol{w}$:



i	~~0~~ ~~1~~ ~~2~~ 3
s	~~0~~ ~~130~~ ~~280~~ 405
$\boldsymbol{w}_1$	130
$\boldsymbol{w}_2$	150
$\boldsymbol{w}_3$	125

The execution appears on the right.

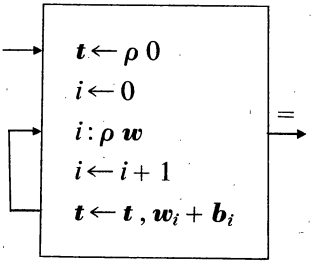
As a further example, the reader may execute the following program for determining the maximum m among the components of $\boldsymbol{w}$:



If $\boldsymbol{w}$ is the vector of weights of $\rho\, \boldsymbol{w}$ boys, if $\boldsymbol{b}$ is the corresponding vector of weights of their baggage, and if $\boldsymbol{t}$ is the vector of total weights, then $\boldsymbol{t}_i$ is calculated as

$$\boldsymbol{t}_i = \boldsymbol{w}_i + \boldsymbol{b}_i$$

Statements 2 through 5 of Program 2.9 perform this addition for all components of $\boldsymbol{w}$ and $\boldsymbol{b}$, using catenation to append the successive components of $\boldsymbol{t}$ as they are computed. The vector $\boldsymbol{t}$ must of course be assigned an initial value before the catenation of statement 5 can be executed. In this initial state the vector $\boldsymbol{t}$ must contain no components and must therefore be of dimension 0. For reasons discussed later, in the treatment of two-dimensional arrays in Chapter 4, the result of applying the dimension function ρ to any scalar is a vector of dimension 0. Hence statement 1 of Program 2.9 (that is, $\boldsymbol{t} \leftarrow \rho\ 0$) gives to $\boldsymbol{t}$ the required initial value.



i	~~1~~ ~~2~~ 3
$\boldsymbol{w}$	130 , 150 , 125
$\boldsymbol{b}$	100 , 75 , 115
$\boldsymbol{t}$	 ~~230~~ ~~230~~ , ~~225~~ 230 , 225 , 240

Program 2.9 The component-by-component sum of vectors

An execution of Program 2.9 is shown for the case where the given values of the arguments $\boldsymbol{w}$ and $\boldsymbol{b}$ are (130 , 150 , 125) and (100 , 75 , 115) respectively. The value of a vector of dimension 0 is denoted by a blank as shown. The following more abbreviated record of the execution is more convenient and, since it is perfectly clear, also acceptable.

i	~~1~~ ~~2~~ 3
$\boldsymbol{w}$	130 , 150 , 125
$\boldsymbol{b}$	100 , 75 , 115
$\boldsymbol{t}$	230 , 225 , 240

(Do Exercise 2.22.)

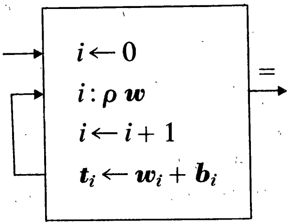
The indexing function $\boldsymbol{x}_i$ (selecting the ith component of $\boldsymbol{x}$) is meaningful only if i is one of the integers $1, 2, 3, \ldots, \rho\ \boldsymbol{x}$. For any other values of i it is therefore undefined and cannot be evaluated. For example, if

$$\boldsymbol{t} = 230 , 225$$

then the statement $z \leftarrow \boldsymbol{t}_3$ cannot be executed. Moreover the statement

$$\boldsymbol{t}_3 \leftarrow 240$$

cannot be executed either. Hence the following program (which might have been considered as a simpler alternative to Program 2.9) is not acceptable and cannot be executed:



A component of a vector is a scalar. A scalar cannot be indexed, that is, if x is a scalar the expression x_i is meaningless, and if $\boldsymbol{y}$ is a vector the expression $\boldsymbol{y}_i$ is meaningful but the expression $(\boldsymbol{y}_i)_1$ is not. If $\boldsymbol{x} = (\rho\ 0), 6$, then $\boldsymbol{x}$ is a vector of dimension 1 and is distinguished from the scalar $y = 6$ in two ways: (1) the vector can be indexed; and (2) $\rho\, \boldsymbol{x}$ is a vector of dimension 1 whose one component is equal to 1, whereas $\rho\, y$ is a vector of dimension 0.

Functions of Vectors

Component-by-component. The need for component-by-component addition of two vectors frequently arises, and this function will be denoted by

$$\boldsymbol{t} \leftarrow \boldsymbol{w} + \boldsymbol{b}$$

It is defined precisely by Program 2.9, except that it is further understood to apply only to vectors $\boldsymbol{w}$ and $\boldsymbol{b}$ of the same dimension. Thus if $\boldsymbol{w} = (130, 150, 125)$ and if $\boldsymbol{b} = (100, 75, 115)$, then $\boldsymbol{t} = (230, 225, 240)$; but if $\boldsymbol{w} = (130, 150, 125)$ and $\boldsymbol{b} = (100, 70, 115, 85)$, then $\boldsymbol{w} + \boldsymbol{b}$ is not defined.

Nevertheless, one of the arguments may be a scalar. For example, if the boys' baggage consists only of packs of the same weight b, then the calculation of the total weight $\boldsymbol{t}$ will be written as

$$\boldsymbol{t} \leftarrow b + \boldsymbol{w}$$

and will imply that $\boldsymbol{t}_i = b + \boldsymbol{w}_i$.

It is useful to apply this component-by-component definition not only to addition but to all other functions in the same way. For example, if $\boldsymbol{t} = (230, 225, 240)$ and if $\boldsymbol{r} = (2, 1, 0.5)$ is the rate per pound to be charged for each of the boys, then the cost for each is given by the vector $\boldsymbol{c}$ as

$$\boldsymbol{c} \leftarrow \boldsymbol{r} \times \boldsymbol{t}$$

and clearly $\boldsymbol{c} = (460\,, 225\,, 120)$. Similarly, if a common rate r is used, then

$$\boldsymbol{c} \leftarrow r \times \boldsymbol{t}$$

For example, if $r = 2$, then $\boldsymbol{c} = (460\,, 450\,, 480)$, and if $\boldsymbol{z} = (25\,, 16\,, 9)$, then $\sqrt{}\,\boldsymbol{z} = (5\,, 4\,, 3)$.

Reduction by a function of two arguments. The summation of all the components of a vector $\boldsymbol{x}$ illustrates another useful type of function. It will be denoted by $+/\,\boldsymbol{x}$ and will be defined as

$$+/\,\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \ldots + \boldsymbol{x}_{(\rho\,\boldsymbol{x})-1} + \boldsymbol{x}_{\rho\,\boldsymbol{x}}$$

where the indicated additions are performed from right to left according to the usual convention.

Again it is useful to extend the notion to all functions of two arguments. For example, $\times/\,\boldsymbol{x}$ denotes the product over all components of $\boldsymbol{x}$. In general, for any function F of two arguments, $F/\,\boldsymbol{x}$ denotes the function $\boldsymbol{x}_1\,F\,\boldsymbol{x}_2\,F\ldots F\,\boldsymbol{x}_{\rho\,\boldsymbol{x}}$ and is called "F-reduction of $\boldsymbol{x}$" or "reduction of $\boldsymbol{x}$ by F" or simply "F over $\boldsymbol{x}$." For example, $+/\,\boldsymbol{x}$ is called "addition-reduction of $\boldsymbol{x}$" or "the sum over $\boldsymbol{x}$."

In the case of subtraction-reduction, the right-to-left order of execution is significant. For example, if $\boldsymbol{x} = 1\,, 4\,, 9\,, 16\,, 25$, then

$$\begin{aligned} -/\,\boldsymbol{x} &= 1 - (4 - (9 - (16 - 25))) \\ &= 15 \end{aligned}$$

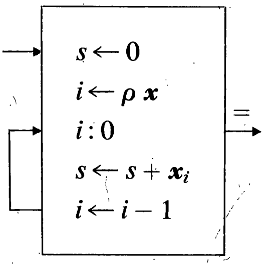
On the other hand, left-to-right execution would yield the result $(((1 - 4) - 9) - 16) - 25) = {}^{-}53$. In general,

$$-/\,\boldsymbol{y} = (\boldsymbol{y}_1 + \boldsymbol{y}_3 + \boldsymbol{y}_5 + \ldots) - (\boldsymbol{y}_2 + \boldsymbol{y}_4 + \boldsymbol{y}_6 + \ldots)$$

Similarly,

$$\div/\,\boldsymbol{y} = (\boldsymbol{y}_1 \times \boldsymbol{y}_3 \times \boldsymbol{y}_5 \times \ldots) \div (\boldsymbol{y}_2 \times \boldsymbol{y}_4 \times \boldsymbol{y}_6 \times \ldots)$$

The following program provides a precise definition of the function $+/\,\boldsymbol{x}$:



Since the index i is initially set to $\rho\, \boldsymbol{x}$ rather than zero and is decreased on each iteration, the components are summed in the required order from right to left.

(Do Exercise 2.23.)

Applications of Vectors

In the foregoing examples the functions on vectors permitted a function on a whole family of variables to be denoted as conveniently as the corresponding function on a single variable. Because of this convenience it is advisable to watch for opportunities to treat a collection of variables as a vector. A few examples will be given in this section. They include points in a plane, points in three-dimensional space, rational numbers, and complex numbers.

Points in space. A point P in a plane having coordinates x and y is often denoted by $P(x, y)$ as shown in Figure 2.10. The two coordinates of the point P can be considered components of a vector $\boldsymbol{p}$ such that $\boldsymbol{p}_1$ is the first coordinate (that is, x) and $\boldsymbol{p}_2$ is the second coordinate.

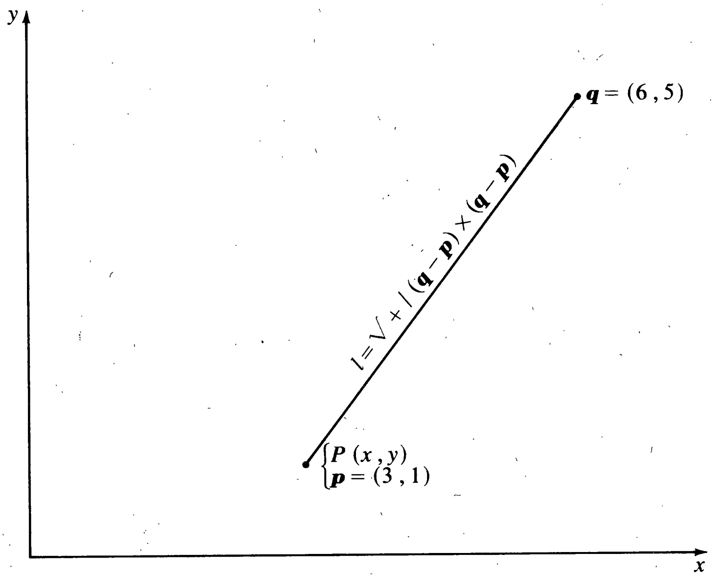


Figure 2.10 Vector representation of points in a plane

If $\boldsymbol{q}$ is the vector of coordinates of a second point in the plane, then clearly the vector

$$\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$$

is the displacement required to move from point $\boldsymbol{p}$ to point $\boldsymbol{q}$. Using the points shown in Figure 2.10, for example, $\boldsymbol{p} = (3\,,1)\,, \boldsymbol{q} = (6\,,5)$, and $\boldsymbol{d} = (3\,,4)$. Furthermore the *length* l of this displacement (that is, the direct distance from $\boldsymbol{p}$ to $\boldsymbol{q}$) is given by

$$l = \sqrt{\ } +/\ \boldsymbol{d} \times \boldsymbol{d}$$

Again in Figure 2.10, $\boldsymbol{d} \times \boldsymbol{d} = (3^2\,,4^2) = (9\,,16)$, and $+/\ \boldsymbol{d} \times \boldsymbol{d} = 9 + 16 = 25$. Finally, $l = \sqrt{\ } 25 = 5$.

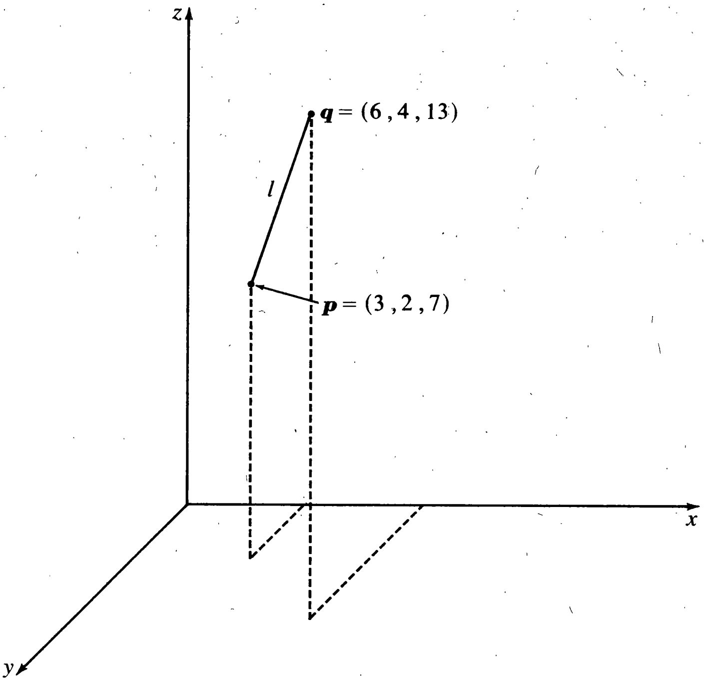


Figure 2.11 Vector representation of points in 3-space

Consideration of points in three-dimensional space reveals some of the power of vector notation, since the expressions for displacement and distance turn out to be identical with the corresponding expressions for points in the plane. The three coordinates of a point in three-dimensional space are of course denoted by a vector $\boldsymbol{p} = (\boldsymbol{p}_1\,, \boldsymbol{p}_2\,, \boldsymbol{p}_3)$ of dimension 3. Thus in Figure 2.11, the displacement is

$$\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p} = (3\,,2\,,6)$$

and

$$l = \sqrt{\ } +/\ \boldsymbol{d} \times \boldsymbol{d} = 7$$

Rational numbers. A rational number n is any number that is expressed as the quotient of two integers i and j, that is, $n = i \div j$. This pair of integers can be treated as a vector $\boldsymbol{r}$ of dimension 2. Thus a vector $\boldsymbol{r}$ whose components are integers represents a rational number n such that

$$n = \boldsymbol{r}_1 \div \boldsymbol{r}_2$$

For example, $\boldsymbol{r} = (3\,,4)$ represents $3 \div 4 = 0.75$, and $\boldsymbol{q} = (2\,,3)$ represents $2 \div 3$. If $\boldsymbol{p}$ represents the product of these two rational numbers, then clearly $\boldsymbol{p} = \boldsymbol{q} \times \boldsymbol{r}$. In the present example, $\boldsymbol{p} = (6\,,12)$. This representation is not unique; the number could also be represented (in lowest terms) by $\boldsymbol{l} = (1\,,2)$, since $\div/\,\boldsymbol{l} = \div/\,\boldsymbol{p}$.

The addition of rationals $\boldsymbol{q}$ and $\boldsymbol{r}$ is somewhat more complicated. If $\boldsymbol{s}$ is the sum, then

$$\boldsymbol{s} = (\,(\boldsymbol{q}_1 \times \boldsymbol{r}_2) + \boldsymbol{q}_2 \times \boldsymbol{r}_1\,), \boldsymbol{q}_2 \times \boldsymbol{r}_2$$

$$\frac{q_1}{q_2} + \frac{r_1}{r_2} = \frac{q_1 \times r_2 + q_2 \times r_1}{q_2 \times r_2}$$

Again using $\boldsymbol{q} = 2\,,3$ and $\boldsymbol{r} = 3\,,4$,

$$\begin{aligned} \boldsymbol{s} &= (\,(2 \times 4) + 3 \times 3\,),\ 3 \times 4 \\ &= (17\,,12) \end{aligned}$$

and the sum of $\frac{3}{4}$ and $\frac{2}{3}$ is $\frac{17}{12}$.

Complex numbers. A complex number n can be represented by a vector $\boldsymbol{c}$ of dimension 2 such that $n = \boldsymbol{c}_1 + \boldsymbol{c}_2 \times \sqrt{^-1}$, where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are real numbers. The addition of two complex numbers represented by $\boldsymbol{c}$ and $\boldsymbol{d}$ is simple:

$$\boldsymbol{s} = \boldsymbol{c} + \boldsymbol{d}$$

Multiplication is slightly more complicated and will be left as an exercise.

Other applications. Suppose that a certain set of plywood panels suffices to make a covered box 2 feet high, 4 feet long, and 3 feet wide. Then the set can be characterized by the numbers 2, 4, and 3. More generally, a group of such sets of panels could be described by a list of the following form:

h	***l***	***w***
2	4	3
1	6	5
4	8	6
1	3	2

where the *i*th row gives the dimensions of the *i*th box. If the successive *columns* of this table are denoted by $\boldsymbol{h}$ (the vector of *heights*), $\boldsymbol{l}$, and $\boldsymbol{w}$, then the volumes $\boldsymbol{v}$ of the set of boxes are given by

$$\boldsymbol{v} = \boldsymbol{h} \times \boldsymbol{l} \times \boldsymbol{w}$$

Similarly, the surface areas $\boldsymbol{s}$ are given by

$$\boldsymbol{s} = 2 \times (\boldsymbol{h} \times \boldsymbol{l}) + (\boldsymbol{h} \times \boldsymbol{w}) + \boldsymbol{l} \times \boldsymbol{w}$$

The ratios $\boldsymbol{r}$ of volumes to surface areas are given by

$$\boldsymbol{r} = \boldsymbol{v} \div \boldsymbol{s}$$

and the set giving the best ratio can be obtained by a program which determines the maximum over the components of $\boldsymbol{r}$.

(Do Exercises 2.24–2.27.)

Programming Techniques

The ability to write good programs, like writing ability of any kind, can be acquired only by practice. Four basic techniques will be presented and illustrated in this section:

1) program reading
2) placing decisions (that is, branches) early in the program
3) utilizing known programs for simple operations within a more complex program
4) programming the main operation *before* the statements that perform the auxiliary operations such as the indexing of vectors

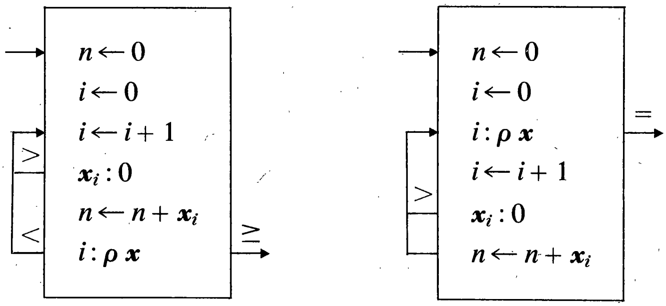
Program reading. To write programs one must first learn to *read* programs critically. This is important not only for understanding the examples provided by completed programs, but also in criticizing and revising attempted programs.

The first step in program reading is learning to perform an accurate execution for any selected initial values of the arguments. The next is to grasp the overall behavior of the program. In Program 2.12 (b), for example, it might be necessary to execute for several different initial values of $\boldsymbol{x}$ to learn that the function determines the sum of the negative components of $\boldsymbol{x}$. A completed program should always be executed to try to discover any subtle flaws in its performance.

Leading decisions. Sequence arrows that cross each other make a program unnecessarily difficult to read and usually betray a poorly organized program. Reordering of the sequence in which the statements are written can usually lead to improvement.

One rule that tends to avoid certain errors and to simplify the branching structure is to place the branch statements early in the program. This is called the use of *leading decisions.*

Consider, for example, Program 2.12 (a), which determines n as the sum of the negative components of $\boldsymbol{x}$. The exit branch occurs last in the program. Execution for the case $\boldsymbol{x} = (3\,, {}^{-}2\,, {}^{-}4\,, 5)$ reveals a flaw—the program does not terminate properly if the last component of $\boldsymbol{x}$ is nonnegative. This kind of error is avoided naturally by placing the exit branch early, as in Program 2.12 (b).



(a) (b)

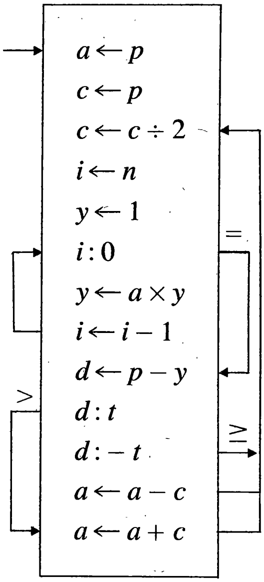
Program 2.12 The sum of the negative components of a vector

Utilizing known programs. A complex program can frequently be built up from simpler known programs. Consider, for example, the problem of determining a as the nth root of the argument p for any positive integer n. Program 2.6 is a solution for the special case of $n = 2$. It can be extended to the present case by replacing the fourth statement with a program segment that determines d as the difference between p and the nth power of a.

2.6
pg 14

The nth power of a can in turn be provided by Program 2.5 (b). Hence a complete program can be constructed by incorporating Program 2.5 in Program 2.6 as shown in Program 2.13. This solution, however, is subject to the limitations indicated in Exercise 2.14.

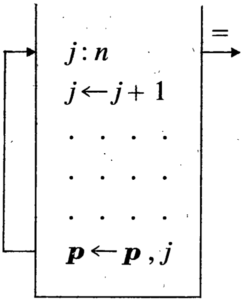
2.5(b)
pg 13



Program 2.13 The *n*th root of *p*

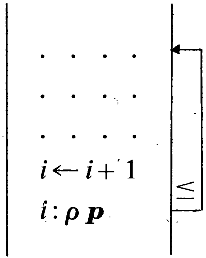
The main operation. It is usually advisable to write the statement for the main operation of a program first and complete other details later. In other words, one begins in the middle of the process, treating, for example, the *i*th component of the vector involved and dealing later with the question of how the index *i* is set initially, modified, and compared to terminate the process.

Consider, for example, the following problem: Compute the vector $\boldsymbol{p}$ which contains as components all the prime numbers not greater than the integer n. For example, if $n = 12$, then $\boldsymbol{p}$ must be the vector $(2, 3, 5, 7, 11)$. The main operation, therefore, is to make a variable j run through the integer values up to n, and to append to $\boldsymbol{p}$ each value of j that proves to be a prime. The overall program therefore includes the following statements:

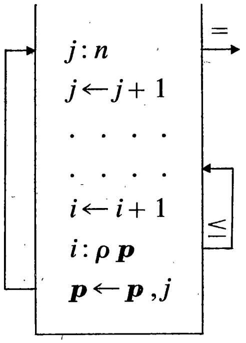


A test of whether j is prime must of course be interposed before the last statement.

The required test is that j must not be divisible by any of the smaller primes, that is, it must not be divisible by any of the present components of $\boldsymbol{p}$. Hence the test for divisibility by $\boldsymbol{p}_i$ must be repeated for each value of i and must therefore be followed by the steps:

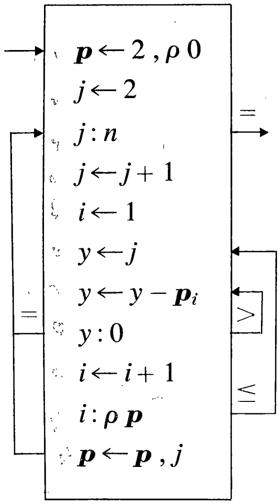


The program thus far therefore appears as



where the two rows of dots indicate a program segment which tests j for divisibility by $\boldsymbol{p}_i$. Divisibility can be tested by repeated subtraction as shown in statements 6 through 8 of the final program (Program 2.14). The accompanying execution for the case $n=9$ illustrates the behavior.

(Do Exercises 2.28–2.31.)

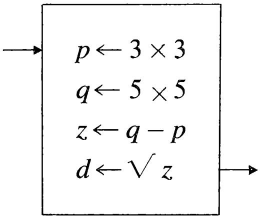


n	9
i	~~1~~ ~~2~~ ~~1~~ ~~1~~ ~~2~~ ~~3~~ ~~1~~ ~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~1~~ ~~1~~ 2
j	~~2~~ ~~3~~ ~~4~~ ~~5~~ ~~6~~ ~~7~~ ~~8~~ 9
y	~~3~~ ~~1~~ ¯~~1~~ ~~4~~ ~~2~~ ~~0~~ ~~5~~ ~~3~~ ~~1~~ ¯~~1~~ ~~5~~ ~~2~~ ~~0~~ ~~1~~ ~~6~~ ~~4~~ ~~2~~ ~~0~~ ~~7~~ ~~5~~ ~~3~~ ~~1~~ ¯~~1~~ ~~7~~ ~~4~~ ~~1~~ ¯~~2~~ ~~7~~ ~~2~~ ¯~~3~~ ~~8~~ ~~6~~ ~~4~~ ~~2~~ ~~0~~ ~~9~~ ~~7~~ ~~5~~ ~~3~~ ~~1~~ ¯~~1~~ ~~9~~ ~~6~~ ~~3~~ 0
$\mathbf{p}$	2 , 3 , 5 , 7

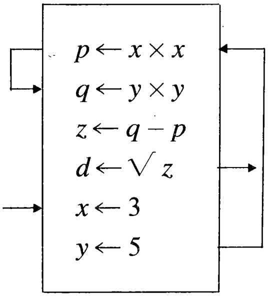
Program 2.14 The vector $\mathbf{p}$ of the primes up to n

Exercises

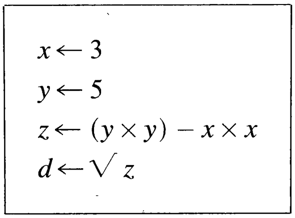
2.1 Execute each of the following programs:



(a)



(b)



(c)

2.2 Execute each of the following programs. (Be sure to observe the right-to-left order.)

(a)

```
$x \leftarrow 3$
$y \leftarrow 5$
$d \leftarrow y \times y$
$z \leftarrow d - x \times x$
$d \leftarrow \sqrt{\ } z$
```

(b)

```
$x \leftarrow 3$
$y \leftarrow 5$
$d \leftarrow y \times y$
$d \leftarrow d - x \times x$
$d \leftarrow \sqrt{\ } d$
```

(c)

```
$a \leftarrow 3$
$b \leftarrow 4$
$x \leftarrow a + b$
$x \leftarrow x \times a - b$
$y \leftarrow a \times a$
$y \leftarrow y - b \times b$
$e \leftarrow x - y$
```

2.3 Execute each of the following programs for $a = 4$ and $b = 9$:

(a)

```
$z \leftarrow a$
$z \leftarrow z \times a + 1$
$z \leftarrow z \times a + 2$
```

(b)

```
$z \leftarrow a$
$a \leftarrow a + 1$
$z \leftarrow z \times a$
$a \leftarrow a + 1$
$z \leftarrow z \times a$
```

(c)

```
$z \leftarrow a \times b$
$z \leftarrow 4 \times z$
```

(d)

```
$y \leftarrow (a + b)$
$z \leftarrow a - b$
$z \leftarrow (y \times y) - (z \times z)$
```

2.4 For each program of Exercise 2.3, write an equivalent one-statement program that is fully parenthesized. For example, 2.3 (a) is equivalent to $z \leftarrow (a \times (a + 1)) \times (a + 2)$. Check each result by actual execution for some chosen values of the arguments.

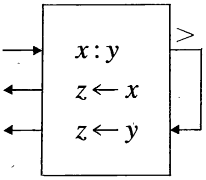
2.5 Rewrite the one-statement programs of Exercise 2.4 using the right-to-left convention to eliminate as many parentheses as possible.

2.6 For each program of Exercise 2.3, state as simply as possible in words what each program does. For example, Program (a) determines z as the product of a and the next two integers following it. Assume that the arguments are restricted to integer values.

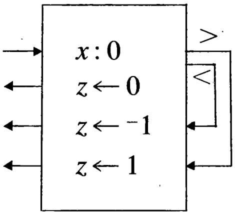
2.7 Write a program for each of the following:
- (a) To determine the length f of a face diagonal of a cube whose edges are of length s.
- (b) To determine the length b of a body diagonal of a cube whose edges are of length s.
- (c) To determine the length b of a body diagonal of a cube whose face diagonal is of length f.

2.8
- (a) Execute your program for Exercise 2.7 (a) for $s = 2$.
- (b) Execute your program for Exercise 2.7 (b) for $s = 2$.
- (c) Execute your program for Exercise 2.7 (c) for $f = 18$.

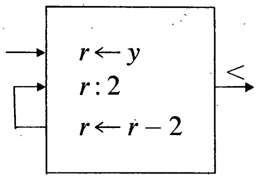
2.9 Execute the following programs for $x = 4$ and $y = 11$:



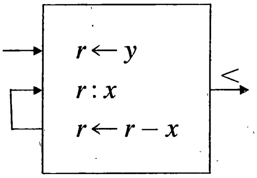
(a)



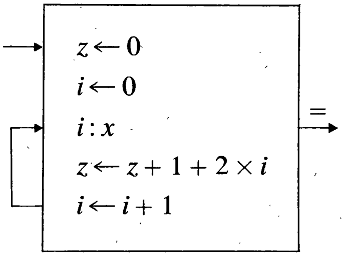
(b)



(c)



(d)



(e)

2.10 Assuming that the arguments x and y are restricted to positive integers, state in words what each program of Exercise 2.9 does. (HINT: Execute each program for values of the arguments selected so as to explore all possible sequences in the program.)

2.11 (a) Determine in how many different orders the statements of Program 2.3 can be executed. Select sets of values of a, b, and c so that the executions exhaust all cases.

(b) Write a program to determine *a* as the absolute value of *x*.

(c) The program of Exercise 2.9 (b) can be modified by the addition of a single statement so as to determine *a* as the absolute value of *x*. Make such a modification.

2.12 (a) Let *a*, *b*, and *c* be the lengths of the three sides of a triangle. Write a program to determine the type of triangle they form, that is, to produce a result *t* which is equal to 1 for scalene, 2 for isosceles, 3 for equilateral.

(b) Extend the program of part (a) to reverse the sign of the result if the triangle is also a right triangle, for example, $t = ^-2$ for an isosceles right triangle.

2.13 Execute Programs 2.5 (a) and (b) for the following cases:

(a) $a = 3, n = 2$

(b) $a = 1, n = 3$

(c) $a = 2, n = 0$

2.14 (a) Execute Program 2.6 for $p = 24$ and $t = 0.25$.

(b) For $p = 0.1$ and $t = 0.05$, execute Program 2.6 far enough to appreciate why it will never terminate.

(c) The defect encountered in part (b) can be corrected by giving *c* a sufficiently large initial value. Write a modification of Program 2.6 that will work for all non-negative values of *p*.

(d) A program to determine the *n*th root of *p* can be obtained by replacing statement 4 of Program 2.6 with a program segment that computes the *n*th power of *a* and subtracts it from *p*. Write such a modification of Program 2.6.

(e) Execute your solution to part (d) for the case $n = 3$, $p = 16$, and $t = 0.5$, and check the resulting value of *a*.

2.15 For $n = 2$, Programs 2.5 and 2.6 define *inverse* functions (that is, square and square root) so that if the result *p* of Program 2.5 is used as the argument *p* of Program 2.6 it will yield (approximately) the original argument *x* used in Program 2.5. Conversely, the result *a* of Program 2.6 used as the argument of Program 2.5 yields the original argument used in Program 2.6.

(a) Use Program 2.5 (b) to check your solution to Exercise 2.14 (a).

(b) Use Program 2.6 to check your solution to Exercise 2.13 (a).

(c) The program required in Exercise 2.14 (d) produces a function (the nth root) that is inverse to the function of Program 2.5 (b) for any value of n. Use Program 2.5 (b) to check your result to Exercise 2.14 (e). (Since t was not small, do not expect very close agreement.)

2.16 (a) Execute Program 2.8 for $k = 48$, and compare the resulting value of the perimeter p with the known value of $2 \times \pi$.

(b) Write a program to approximate $2 \times \pi$ by *circumscribed* polygons.

(c) Write a program (combining the programs of parts (a) and (c)) to determine both upper and lower bounds on $2 \times \pi$ and to terminate when the difference between them is less than a given tolerance t.

2.17 Write a program which determines q as the integral part of the quotient $x \div y$ and r as the remainder, where the arguments x and y are restricted as follows:

(a) x and y are both positive.

(b) y is nonzero.

2.18 Write a program for each of the following:

(a) To determine a as the value of $!n$ (that is, factorial n, defined as $1 \times 2 \times 3 \times \ldots \times n$), for nonnegative integers n. Ensure that the program gives the correct value of $!0$, which is defined as 1.

(b) To determine s as the sum of the first n positive integers.

(c) To determine q as the sum of the squares of the first n positive integers.

2.19 (a) Execute your solution to Exercise 2.18 (b) for the first few values of n and compare the values of s with the corresponding values of s in the program

$$s \leftarrow (n \times n + 1) \div 2$$

(b) Execute your solution to Exercise 2.18 (c) for a few values and compare with the program

$$q \leftarrow (n \times (n + .5) \times n + 1) \div 3$$

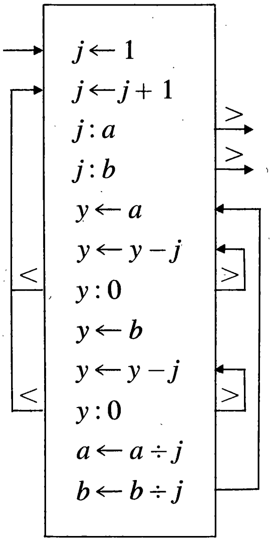
2.20 (a) Execute the accompanying program for the following cases:

(i) $a = 30, b = 42$

(ii) $a = 15, b = 5$

(b) Assuming that the variables a and b together represent the rational number $a \div b$, state in words what the program does.

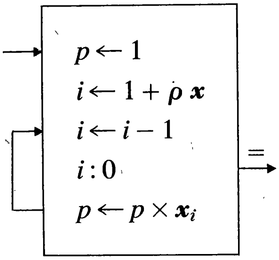
(c) Write a program which determines the greatest common divisor of the integer arguments m and n.



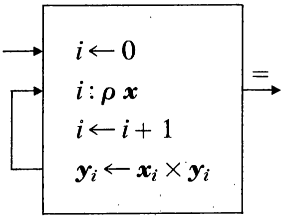
2.21 (a) Write a program to determine z as a function of x such that $z = x^{m \div n}$. (HINT: See Program 2.5 and Exercise 2.14 (d) and note that $x^{m \div n} = (\sqrt[n]{x})^m$.)

(b) Execute the program of part (a) for the case $x = 10$, $m = 3$, $n = 4$.

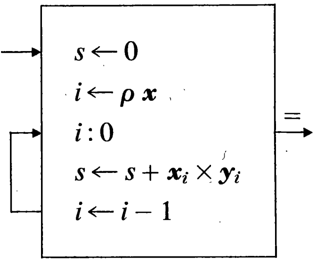
2.22 Execute the following programs for the case $\boldsymbol{x} = (1, 2, 3, 4)$ and $\boldsymbol{y} = (4, 3, 2, 1)$.



(a)



(b)



(c)

2.23 Write one-statement programs equivalent to those of Exercise 2.22. For example, $p \leftarrow \times / \boldsymbol{x}$ is equivalent to 2.22 (a).

2.24 A rectangular box is to be constructed. The materials available permit any one of several specified combinations of length, width, and height. Write a program which will determine the set of dimensions that give a box having the maximum ratio of volume to surface area. In writing the program—

(a) First prepare an interpretation table to show your choice of symbols for the variables involved.
(b) Use vector operations where possible.
(c) Execute the program for some simple case.

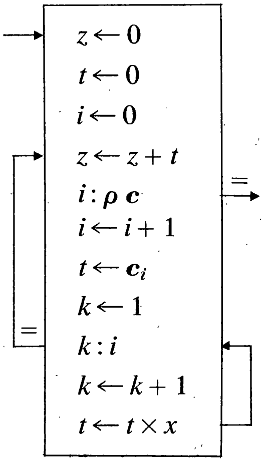
2.25 The following programs perform a pair of functions which should be familiar from elementary arithmetic. Identify these functions. (HINT: Execute the programs for several initial values. If necessary, refer to the section devoted to applications of vectors, page 25.) The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are each of dimension 2 (that is, $\rho\,\boldsymbol{x} = \rho\,\boldsymbol{y} = 2$):

(a) $\boldsymbol{z} \leftarrow ((\boldsymbol{x}_1 \times \boldsymbol{y}_2) + \boldsymbol{x}_2 \times \boldsymbol{y}_1)\ , \boldsymbol{x}_2 \times \boldsymbol{y}_2$
(b) $\boldsymbol{z} \leftarrow \boldsymbol{x} \times \boldsymbol{y}$

2.26 As in Exercise 2.25, identify the pair of familiar functions performed by the following programs:

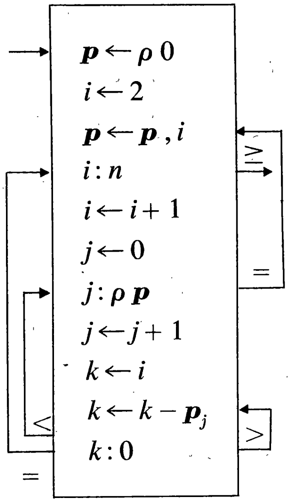
(a) $\boldsymbol{z} \leftarrow \boldsymbol{x} + \boldsymbol{y}$
(b) $\boldsymbol{z} \leftarrow ((\boldsymbol{x}_1 \times \boldsymbol{y}_1) - \boldsymbol{x}_2 \times \boldsymbol{y}_2)\ , (\boldsymbol{x}_1 \times \boldsymbol{y}_2) + \boldsymbol{x}_2 \times \boldsymbol{y}_1$

2.27 State in words the function performed by the following program:



```
z ← 0
t ← 0
i ← 0
z ← z + t
i : ρ c        =
i ← i + 1
t ← c_i
k ← 1
k : i
k ← k + 1
t ← t × x
```

2.28 (a) The accompanying program produces a vector $\boldsymbol{p}$ as a function of n. State in words what the program does. (Execute for a few values of n if necessary.)



(b) Modify Program 2.14 so that it will determine $\boldsymbol{p}$ as the vector of the first n primes rather than as the primes up to n.

2.29 (a) If $\boldsymbol{x} \leftarrow 1 , (-2) , 3 , 4 , (-5) , 6$, then

$$\begin{array}{lll} \boldsymbol{x}_1 = 1 & \boldsymbol{x}_2 = -2 & \boldsymbol{x}_3 = 3 \\ \boldsymbol{x}_4 = 4 & \boldsymbol{x}_5 = -5 & \boldsymbol{x}_6 = 6 \end{array}$$

Show that if $\boldsymbol{y} \leftarrow 1 , -2 , 3 , 4 , -5 , 6$, then $\boldsymbol{y}$ and $\boldsymbol{x}$ are *not* equivalent and in fact

$$\begin{array}{lll} \boldsymbol{y}_1 = 1 & \boldsymbol{y}_2 = -2 & \boldsymbol{y}_3 = -3 \\ \boldsymbol{y}_4 = -4 & \boldsymbol{y}_5 = 5 & \boldsymbol{y}_6 = 6 \end{array}$$

(HINT: Be careful to observe the exact right-to-left sequence in performing the successive catenations (denoted by commas) and negations.)

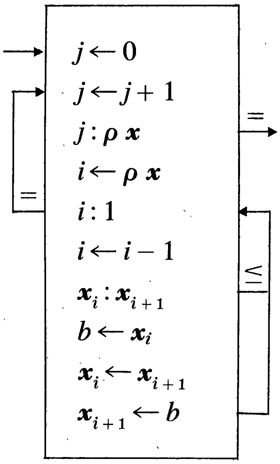
(b) Show that the components of the vector

$$\boldsymbol{a} \leftarrow 3 , -4 , -5 , -7 , -8 , -1$$

are alternately positive and negative.

(c) Write a statement using only raised negative signs and catenation to specify a vector $\boldsymbol{z}$ which is equal to the vector $\boldsymbol{x}$ of part (a).

2.30 (a) Execute the accompanying program for the following cases:



(i) $\boldsymbol{x} = 6, 2, 1, 9$
(ii) $\boldsymbol{x} = 6, 2, 3, 5, 2, 8, 1$

(b) State in words what the program does to the vector $\boldsymbol{x}$.

(c) Write a program which determines $\boldsymbol{d}$ as the vector containing all the *distinct* components of a vector $\boldsymbol{a}$. For example, if $\boldsymbol{a} = 3, 5, 5, 9, 12, 12, 12$, then $\boldsymbol{d} = 3, 5, 9, 12$. Assume that the components of $\boldsymbol{a}$ are already in ascending order.

(d) Execute your program for the case

$$\boldsymbol{a} = 3, 5, 5, 9, 12, 12, 12$$

(e) Modify the program developed in part (c) so that it will work for a vector $\boldsymbol{a}$ whose components are in arbitrary order. (HINT: Use the program of part (a) to arrange the components of $\boldsymbol{a}$ in ascending order.)

(f) Show that the program of part (a) produces the same result and is faster to execute if statement 5 is replaced with $i : j$.

2.31 (a) Let the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be the plane coordinates of the vertices of a triangle. Write a program to determine the same result as in Exercise 2.12 (a), using $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ as arguments.

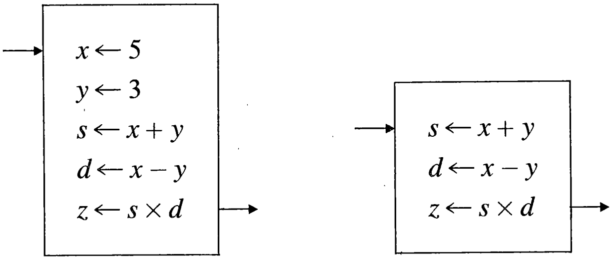
(b) Repeat part (a) assuming that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ represent points in three dimensional space (that is, $\rho\, \boldsymbol{a} = \rho\, \boldsymbol{b} = \rho\, \boldsymbol{c} = 3$).

Chapter Three

Functions

Definition of Functions

A program such as Program 3.1 (a), in which the value of each variable is specified, has no arguments and the result z is assigned a single fixed value. In Program 3.1 (b), on the other hand, the values of the variables x and y are not prescribed and the program can be viewed as a rule for assigning a specific value to z for each specific pair of values assigned to x and y; the program therefore defines a function of the two arguments x and y.



```
x ← 5
y ← 3
s ← x + y
d ← x − y
z ← s × d
```

(a)

```
s ← x + y
d ← x − y
z ← s × d
```

(b)

Program 3.1 Functions of no arguments and of two arguments

The arguments in a program are easily identified as the variables to which initial values are not assigned within the program. The identification of the result variables is less obvious and is, in fact, rather arbitrary. Program 3.1 (b), for example, produces three results: s (the sum of the arguments), d (their difference), and z (the product $s \times d$). It therefore represents three functions of x and y. Usually, however, one of the result variables is singled out as the result of interest and is called the *resultant*; the others are then considered "intermediate" results.

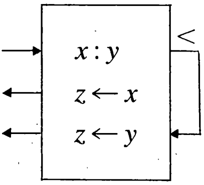
A program therefore provides a means for defining new functions. Once a suitable name or symbol is associated with a program, the function described by that program can be used in other programs as freely as the basic functions already defined. This provides a powerful means for extending the set of functions available for writing programs.

Consider, for example, Program 3.2, which determines z as the maximum of x and y. The expression

$$z \leftarrow x \lceil y$$

above the program is the name assigned to the function described by the program. The symbol $\lceil$ is therefore assigned to the *maximum* function. It is necessary, however, to include the variables z, x, and y in the name so as to make clear which variable in the program represents the resultant, which represents the first argument, and which represents the second argument.

$$z \leftarrow x \lceil y$$

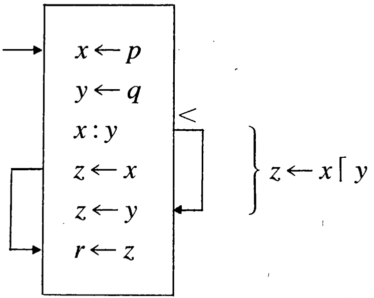


Program 3.2 Definition of the maximum function $\lceil$

A function such as $\lceil$ is assumed to apply not only to the particular variables x and y used in its definition, but equally to any pair of variables whatever. Thus the statement

$$r \leftarrow p \lceil q$$

implies that r is specified as the maximum of p and q. More precisely, it is equivalent to the following program,



in which statements 3 through 5 represent the function previously named: $z \leftarrow x \lceil y$.

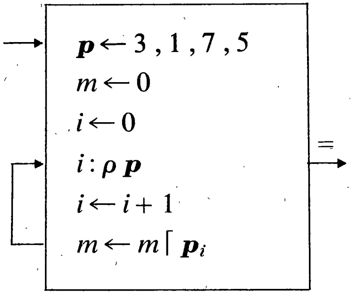
Thus Program 3.3 together with the program for $z \leftarrow x \lceil y$ (Program 3.2) would be executed as follows:

$\mathbf{p}$	3 , 1 , 7 , 5	x	0	3	3	7
m	$\cancel{0}\ \cancel{3}\ \cancel{3}\ \cancel{7}\ 7$	y	3	1	7	5
i	$\cancel{0}\ \cancel{1}\ \cancel{2}\ \cancel{3}\ 4$	z	3	3	7	7
$\mathbf{p}_i$	$\cancel{3}\ \cancel{1}\ \cancel{7}\ 5$					

The successive columns on the right each represent an execution for one of the cases prescribed by the step

$$m \leftarrow m \lceil \mathbf{p}_i$$

in Program 3.3.



Program 3.3 Use of the maximum function

In Program 3.2 the variables x, y, z which appear in the name of the function it defines are simply dummy variables which indicate the role (for example, the first argument) played by the variables occurring in the program. Although it may seem strange that the assignment of a name such as $z \leftarrow x \lceil y$ then permits the use of statements such as $r \leftarrow p \lceil q$ and $m \leftarrow m \lceil \mathbf{p}_i$, the situation is actually no different from that assumed for the basic functions such as + and ×. They are assumed to apply to any set of variables (for example, $r \leftarrow p + q$; $m \leftarrow m \times \mathbf{p}_i$, and so forth) even though their definition for one unfamiliar with them would have to be couched in terms of specific variables such as x, y, and z. Moreover, when one encounters the symbol + in a statement he must (except for small integer arguments) turn aside to execute an algorithm for it, namely, the addition procedure learned in elementary school.

Any new function of two arguments that has been defined by a program and appropriately named can therefore be used afterward in any and all of the ways that a basic function such as addition can be used. For example, $+/\boldsymbol{x}$ denotes the sum $\boldsymbol{x}_1+\boldsymbol{x}_2+\ldots+\boldsymbol{x}_{\rho x}$ and, analogously, $\lceil/\boldsymbol{x}$ denotes $\boldsymbol{x}_1\lceil\boldsymbol{x}_2\lceil\ldots\lceil\boldsymbol{x}_{\rho x}$. Consequently the program

$$\boldsymbol{p}\leftarrow 3\,,1\,,7\,,5$$
$$m\leftarrow\lceil/\boldsymbol{p}$$

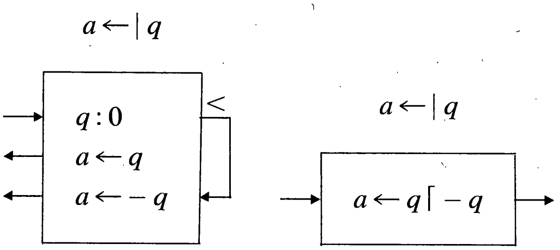
(Do Exercises 3.1 – 3.2.)

is equivalent to Program 3.3.

Naming Functions

A function of one argument is called a *monadic* function, and a function of two arguments is called *dyadic*. To name a dyadic function, the symbol for the function will be placed between the arguments. This agrees with the form for the basic dyadic arithmetic functions + and × and was done for the maximum function in Program 3.2. Moreover, the symbol used for a function will be a special symbol such as $\lceil$, a Greek letter, or an uppercase Roman letter.

Monadic functions are named in a similar way, with the symbol for the function being placed before the argument. This agrees with the form normally used for negation, that is, the symbol − is placed before the argument. The familiar notation for absolute value, however, does not follow this pattern, since the symbol both precedes and follows the argument, thus: $|x|$. This notation will therefore be replaced with $|\,x$, so that the symbol for the function precedes the argument. The absolute value function can be formally defined by Program 3.4 (a) or by Program 3.4 (b).



(a) (b)

Program 3.4 Definitions of the absolute value function |

This use of the same symbol to denote both a dyadic function (for example, $x-y$) and a related monadic function ($-y$) introduces no ambiguity, since the context determines the interpretation. For example, in the expression $x + y - z$ the symbol $-$ represents a dyadic function, whereas in the expression $x \times - z$ it necessarily represents a monadic function. Any symbol can therefore be used unambiguously to denote both a dyadic function and a (not necessarily related) monadic function. For example, the symbol $|$ used for the monadic function *absolute value* will also be used to denote the dyadic function *residue* to be defined in the following section. This double use of some symbols gives no difficulty and helps to keep the symbols required down to a reasonable number.

Two functions are said to be *equivalent* if they yield the same result for any chosen values of their arguments. Equivalence will be denoted by the symbol $\equiv$. For example:

$$(x+1) \times x + 1 \equiv (x \times x) + (2 \times x) + 1$$

Some Basic Functions

The method for defining new functions can now be used to extend the small set of basic functions adopted in Chapter 2; each new function can be defined by a program which employs only the basic functions or previously defined functions. Consequently *the eventual definition of each function is made in terms of a small and simple set of familiar functions.* All notation and all functions defined in the text are listed in Appendix D.

Maximum, minimum, negation, and absolute value. *Negation* is a monadic function denoted by the symbol $-$ and defined in terms of subtraction as follows:

$$- x \equiv 0 - x$$

Maximum is the dyadic function denoted by $\lceil$ and defined by Program 3.2. *Minimum* is denoted by $\lfloor$. It could be defined by a similar program, but (as the reader should verify) it can be defined in terms of maximum and negation as follows:

$$x \lfloor y \equiv - (- x) \lceil - y$$

Absolute value is a monadic function denoted by $|$ and defined by

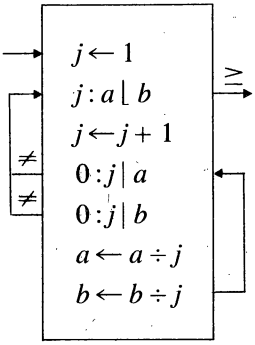
$$| x \equiv x \lceil - x$$

(Do Exercises 3.3–3.4.)

Residue. The *residue of n modulo d* is denoted by $d|n$ and is defined for all values of d and n, except for $d = 0$, as the *nonnegative* remainder obtained on dividing n by d. More precisely, if r is the residue of n modulo d, then $0 \leq r < | d$, and $n = r + q \times d$, where q is some integer. Some sample values of the residue function are shown below:

d	n	$d \mid n$
3	7	1
3	8	2
3	9	0
3	$^{-}7$	2
4	19	3
4	$^{-}19$	1

A simple use of the residue function occurs in reducing a rational number to lowest terms. If a and b are two integers representing the rational number $a \div b$, then Program 3.5 reduces them to lowest terms. The process is simply to divide by j if the residues $j \mid a$ and $j \mid b$ are both zero, and is illustrated by the accompanying execution for the case where $a = 84$ and $b = 140$.

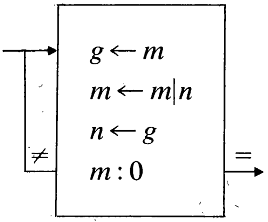


a	~~84~~ ~~42~~ ~~21~~ 3
b	~~140~~ ~~70~~ ~~35~~ 5
j	~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ ~~6~~ 7

Program 3.5 Reduction of the rational number $a \div b$ to lowest terms

The residue function is also used in a very efficient method for finding the greatest common divisor of two integers m and n, the so-called Euclidean algorithm. Any factor common to m and n is also a factor of the remainder obtained in dividing n by m, that is, $m \mid n$. Hence the *gcd* of m and n is also the *gcd* of m and $m|n$. The process can be repeated by obtaining the residue $(m|n)|m$, and so on, as shown explicitly in Program 3.6 and as illustrated in the accompanying execution for $m = 30$ and $n = 42$.

$g \leftarrow m A n$



m	~~30~~ ~~12~~ ~~6~~ 0
n	~~42~~ ~~30~~ ~~12~~ 6
g	~~30~~ ~~12~~ 6

Program 3.6 The Euclidean algorithm for the greatest common divisor of m and n

It should also be noted that $1|x$ is the fractional part of x and that $x - 1|x$ is the integral part of x. The reader can also verify the following relation between the residue function and the monadic function denoted by the same symbol, that is, the absolute-value function:

$$|x \equiv (2 \times x)|x$$

(Do Exercises 3.5–3.9.)

The factorial function. The factorial function is denoted by $!\,n$ and is defined for positive integral values of n as the product of the positive integers up to and including n. For example, $!1 \equiv 1$; $!2 \equiv 2$; $!3 \equiv 6$; and $!8 \equiv 40320$. The value of $!0$ is defined to be 1 so that the obvious identity

$$!\,n \equiv n \times !\,n - 1$$

will hold even for the case $n = 1$, that is,

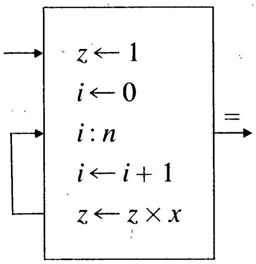
$$!\,1 \equiv 1 \times !\,0$$

The factorial function is defined only for nonnegative integral values of the argument. It is more commonly denoted by $n!$, but $!\,n$ will be used here to obey the convention of placing the symbol for a monadic function *before* its argument.

The exponential function. The function $x E n$ defined by Program 3.7 will be recognized as the function more commonly denoted by x^n and called "x to the power n" or "x to the exponent n." The program defines $x E n$ only for integral exponents, since it never terminates if n is not an integer. The definition was extended to the case $n = 0.5$ (that is, square root) by Program 2.6, and to the case $n = 1 \div m$ by the more general program required in Exercise 2.14. Since $x E (p \div q) \equiv (x E p) E (1 \div q)$ (that is, $x^{p \div q} \equiv (x^p)^{1 \div q}$), the function can obviously be extended to any rational exponent $p \div q$.

Since any irrational number r can be approximated as closely as desired by a rational number, the function can be extended to any exponent r. For example, since $\pi = 3.14159\ldots$ is approximately $\frac{22}{7}$, then x^π is approximately $(x^{22})^{1 \div 7}$.

$$z \leftarrow x\,E\,n$$



Program 3.7 Exponentiation (for integral n only)

The general exponential function defined for all exponents r will be denoted by $x * r$. If r is an integer, then $x * r$ is equivalent to the function $x\,E\,r$ defined by Program 3.7. Since $x * .5$ denotes the square root of x, the special symbol $\sqrt{}$ will no longer be used formally for square root.

The familiar exponential notation x^n violates the convention adopted for denoting a dyadic function; it has, in fact, *no symbol* for the function, which it indicates only by the raised position of the second argument. There are important advantages to using the explicit symbol $*$, which will become clear in the discussion of the fundamental properties of functions. A further advantage is that all extensions to vectors automatically apply to functions denoted in the standard form. Thus $\boldsymbol{x} * \boldsymbol{n}$; $p * \boldsymbol{n}$; $\boldsymbol{x} * q$; and $*/\boldsymbol{y}$ are all meaningful and useful expressions. For example, if $\boldsymbol{x} \equiv (7, 6, 5, 4)$; $\boldsymbol{n} \equiv (0, 1, 2, 3)$; $p \equiv 8$; $q \equiv 2$; and $\boldsymbol{y} \equiv (3, 2, 1, 0)$, then

$$\boldsymbol{x} * \boldsymbol{n} \equiv (7 * 0), (6 * 1), (5 * 2), (4 * 3) \equiv 1, 6, 25, 64$$
$$p * \boldsymbol{n} \equiv (8 * 0), (8 * 1), (8 * 2), (8 * 3) \equiv 1, 8, 64, 512$$
$$\boldsymbol{x} * q \equiv (7 * 2), (6 * 2), (5 * 2), (4 * 2) \equiv 49, 36, 25, 16$$

and

$$*/\boldsymbol{y} \equiv 3 * 2 * 1 * 0 \equiv 9$$

However, because of its familiarity, the notation x^n will continue to be used informally at times.

(Do Exercises 3.10–3.13.)

Relational functions. Any relation (such as $x \leq y$) is either true or false, and the relation can therefore be considered a function whose resultant has one of the two values, "true" or "false". Thus the statement

$$u \leftarrow x \leq y$$

gives to u the value "true" if x is less than or equal to y, and the value "false" otherwise. The value "false" will be represented by the number 0 and the value "true" by the number 1. Thus if $x = 6$ and $y = 8$, then u takes on the value 1. A variable such as u which takes on only the values 0 and 1 is called a *logical* variable.

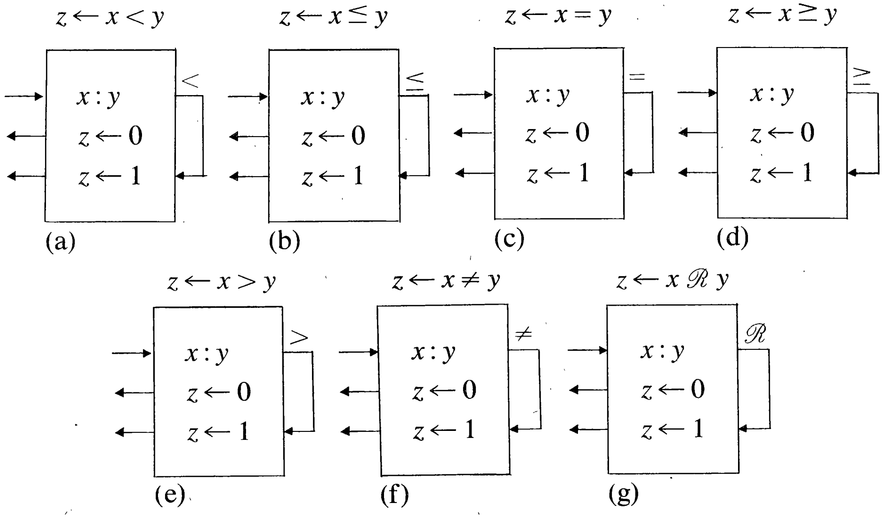
The relational functions are defined formally by Programs 3.8 (a) through (f) and summarized in Program 3.8 (g). They extend to vectors in the usual way. For example, if

$$\boldsymbol{x} \equiv (6,3,7,{}^{-}4) \text{ and } \boldsymbol{y} \equiv (2,3,6,2)$$

then

$$\begin{aligned} \boldsymbol{x} < \boldsymbol{y} &\equiv (0,0,0,1) \\ \boldsymbol{x} \leq \boldsymbol{y} &\equiv (0,1,0,1) \\ \boldsymbol{x} = \boldsymbol{y} &\equiv (0,1,0,0) \\ \boldsymbol{x} \neq \boldsymbol{y} &\equiv (1,0,1,1) \\ \boldsymbol{x} < 0 &\equiv (0,0,0,1) \end{aligned}$$

The logical vector that results from a relational function proves most useful in the compression function to be defined in the following section.



Program 3.8 Definition of the relational functions $\mathscr{R}$

Order reversal and compression. It is convenient to be able to reverse the order of the components of a vector. This can be done by the *reversal function* ⌽. Thus if $\boldsymbol{x} \equiv 3, 1, 6, 4, 2$, then

$$\text{⌽}\ \boldsymbol{x} \equiv 2, 4, 6, 1, 3$$

The formation of a vector $\boldsymbol{y}$ by the selection of certain components from a second vector $\boldsymbol{x}$ will be called *compression* and will be denoted by

$$\boldsymbol{y} \leftarrow \boldsymbol{u} / \boldsymbol{x}$$

where $\boldsymbol{u}$ is a logical vector of the same dimension as $\boldsymbol{x}$. The zeros in $\boldsymbol{u}$ indicate which components of $\boldsymbol{x}$ are to be suppressed. Thus if

$$\boldsymbol{x} \equiv 9, 8, 7, 6, 5, 4, 3$$

and

$$\boldsymbol{u} \equiv 0, 1, 1, 0, 0, 0, 1$$

then

$$\boldsymbol{u} / \boldsymbol{x} \equiv 8, 7, 3$$

Similarly,

$$(1, 0, 1, 0, 1, 0, 1) / \boldsymbol{x} \equiv 9, 7, 5, 3$$

The dimension of the vector $\boldsymbol{u} / \boldsymbol{x}$ is of course equal to the number of 1's in $\boldsymbol{u}$, that is, $\rho(\boldsymbol{u} / \boldsymbol{x}) \equiv +/\ \boldsymbol{u}$.

Compression proves very useful, especially when the logical vector itself is determined by some relational function applied to the integer vector to be defined next.

The vector of integers. The monadic function ι is defined as follows[†]: $\iota\ n$ denotes a vector of dimension n whose ith component is equal to i. Thus

$$\begin{aligned} \iota 3 &\equiv 1, 2, 3 \\ \iota 5 &\equiv 1, 2, 3, 4, 5 \\ \times/\ \iota n &\equiv !\ n \\ +/\ \iota n &\equiv 0.5 \times n \times n + 1 \end{aligned}$$

The last of the preceding identities is the well-known expression for the sum of the first n integers.

The combined uses of the relational functions, compression, and the integer vector will now be illustrated by a number of examples

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[†] ι is the Greek letter corresponding to the Roman i. It is spelled and pronounced *iota*.

which can be verified by choosing some sample values of the arguments:

$(\boldsymbol{x} \geq 0) / \boldsymbol{x}$	selects all nonnegative components of $\boldsymbol{x}$.
$(j \geq \iota\, n) / \boldsymbol{x}$	selects the first j components of $\boldsymbol{x}$.
$(\text{⌽}\, j \geq \iota\, n) / \boldsymbol{x}$	selects the last j components of $\boldsymbol{x}$.
$(j \neq \iota\, n) / \boldsymbol{x}$	suppresses the jth component of $\boldsymbol{x}$.
$(\boldsymbol{x} \neq \lceil / \boldsymbol{x}) / \boldsymbol{x}$	selects all components not equal to the maximum.
$(0 = 3 \mid \boldsymbol{x}) / \boldsymbol{x}$	selects all components of $\boldsymbol{x}$ which are divisible by 3.
$(0 = \boldsymbol{u}) / \boldsymbol{x}$	selects all components of $\boldsymbol{x}$ *not* selected by $\boldsymbol{u}$.
$0 = \iota\, n$	is a vector of n *zeros*.
$0 \neq \iota\, n$	is a vector of n *ones*.

Finally, $\iota\, 0$ clearly denotes a vector of dimension zero, which has no components; it can be used instead of $\rho\, 0$. Similarly, the expression $x \times \iota\, 1$ can be used instead of the expression $(\rho\, 0), x$ to denote a vector of dimension 1 whose one component has the value x.

The argument of the function ι can be either a scalar or a vector of dimension 1. For example, if $\boldsymbol{x} \equiv (7, 8, 9)$, then

$$\iota\, \rho\, \boldsymbol{x} \equiv \iota\, 3 \equiv 1, 2, 3$$

Reduction over vectors of dimensions 0 and 1. Reduction by some dyadic function is frequently applied to a vector which is itself obtained by compression. For example, $+/\ (\boldsymbol{x} \geq 0) / \boldsymbol{x}$ yields the sum of all the nonnegative components of $\boldsymbol{x}$, and $\times/\ (j \geq \iota\, n) / \boldsymbol{x}$ yields the product over the first j elements of $\boldsymbol{x}$.

The vector determined by compression can obviously be of any dimension, including 0 and 1. Since the original definition of reduction involved placing the reducing function between the components (for example, $+/\boldsymbol{x} \equiv \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3 + \ldots + \boldsymbol{x}_{\rho\, \boldsymbol{x}}$), it does not apply to dimensions 0 and 1. The definition will be extended to cover these cases in a simple manner that gives consistent results.

If $\boldsymbol{x}$ is split into two parts $\boldsymbol{u} / \boldsymbol{x}$ and $(0 = \boldsymbol{u}) / \boldsymbol{x}$, then the sum over $\boldsymbol{x}$ must equal the total of the sums over the parts. Hence

$$+/\, \boldsymbol{x} \equiv (+/\ (0 = \boldsymbol{u}) / \boldsymbol{x}) + (+/\, \boldsymbol{u} / \boldsymbol{x})$$

For example, if $\boldsymbol{x} \equiv 2, 3, 4, 5, 6$ and $\boldsymbol{u} \equiv 0, 1, 0, 1, 0$ then $+/\, \boldsymbol{x} \equiv 20$; $+/\, \boldsymbol{u} / \boldsymbol{x} \equiv 8$; $+/\ (0 = \boldsymbol{u}) / \boldsymbol{x} \equiv 12$; and the relation is satisfied. However, if $\boldsymbol{u} \equiv 0, 0, 1, 0, 0$, then $+/\ (0 = \boldsymbol{u}) / \boldsymbol{x} \equiv 2 + 3 + 5 + 6 \equiv 16$ and it is clear that the sum over the single-component vector $\boldsymbol{u} / \boldsymbol{x}$ should be defined as the value of that single component—in this case, 4. This is clearly desirable for dyadic functions other than +, and the reduction of any single-component vector will therefore be defined to be that single component.

The case of a vector of dimension 0 arises if $\boldsymbol{u} \equiv 0\,,0\,,0\,,0\,,0$. Since $+/\boldsymbol{x} \equiv (+/\,(0 = \boldsymbol{u})\,/\,\boldsymbol{x}) + (+/\,\boldsymbol{u}\,/\,\boldsymbol{x})$ and $(0 = \boldsymbol{u})\,/\,\boldsymbol{x} \equiv \boldsymbol{x}$, it is clear that the sum over a vector of dimension 0 should be defined as 0. The reason, of course, is that 0 added to any quantity z yields z.

In the corresponding situation for reduction by a product

$$\times/\,\boldsymbol{x} \equiv (\times/\,(0 = \boldsymbol{u})\,/\,\boldsymbol{x}) \times (\times/\,\boldsymbol{u}\,/\,\boldsymbol{x})$$

and it is clear that the product over a vector of dimension 0 must be defined as 1 rather than 0, since $z \times 1 \equiv z$ for any z.

A value r such that $z\,F\,r \equiv z$ for any value of z is called an *identity* element of the function F. For + the identity element is 0 and for × it is 1. Hence the value of $F/\,\iota\,0$ (that is, the F-reduction of a vector of dimension 0) will be defined as the identity element of the function F. The identity element of the minimum function $\lfloor$ is clearly an element which is larger than any specified number. It is called *infinity* and denoted by ∞. Hence

$$+/\,\iota\,0 \equiv 0 \qquad \times/\,\iota\,0 \equiv 1 \qquad \div/\,\iota\,0 \equiv 1 \qquad -/\,\iota\,0 \equiv 0$$
$$\lfloor/\,\iota\,0 \equiv \infty \qquad \lceil/\,\iota\,0 \equiv -\infty \qquad */\,\iota\,0 \equiv 1$$

It is interesting to note that the foregoing definition for $\times/\,\iota\,0$ yields the correct result for the case $n = 0$ in the identity $!\,n \equiv \times/\,\iota\,n$, that is, $!\,0 \equiv 1$. The residue $x|y$ is an example of a function that does not have an identity element.

(Do Exercises 3.14–3.15.)

Fundamental Properties of Functions

All readers will be familiar with the following identities for the elementary arithmetic functions *addition* and *multiplication*:

$x + y \equiv y + x$	+ is commutative
$x \times y \equiv y \times x$	× is commutative
$x + (y + z) \equiv (x + y) + z$	+ is associative
$x \times (y \times z) \equiv (x \times y) \times z$	× is associative
$x \times (y + z) \equiv (x \times y) + (x \times z)$	× distributes over +

Most readers will also be familiar with the descriptive phrase appended to each identity. In any case, it is easy to deduce from the foregoing list the general definitions of the terms *commutative*, *associative*, and *distributive*. Thus:

1) A dyadic function F is *commutative* if and only if $x F y \equiv y F x$ for all values of x and y.
2) A dyadic function F is *associative* if and only if $x F (y F z) \equiv (x F y) F z$ for all values of x, y, and z.
3) A dyadic function F *distributes* over a dyadic function G if and only if $x F (y G z) \equiv (x F y) G (x F z)$ for all values of x, y, and z.

It is always helpful to understand the reasons for the choice of new mathematical terms encountered. Thus *commute* means to interchange two things; *associative* suggests that the terms in the expression can be associated (by parenthesizing) in any manner; *distributive* suggests that the effect of one function can be distributed over both arguments of the second function.

It is difficult to grasp the full significance of commutativity, associativity, and distributivity if they are applied only to the basic arithmetic functions, for this application simply gives pretentious names to identities already familiar and yields no new information. More insight (and many useful new identities) can be gained by examining the corresponding properties of less familiar functions such as $\lceil$, $\lfloor$, and $*$. The maximum function will be used for illustration.

Commutativity. The maximum function can be shown to be commutative by comparing the expressions $p \lceil q$ and $q \lceil p$ for the two possible cases $p < q$ and $p \geq q$ as shown in Table 3.9. If $p < q$, then execution of the defining program (Program 3.2) shows that $p \lceil q = q$. A similar execution shows that $q \lceil p = q$. Execution for the case $p \geq q$ yields $p \lceil q = p$ and $q \lceil p = p$. Hence $p \lceil q = q \lceil p$ for all values of p and q; that is, the function $\lceil$ is commutative.

Case	$p \lceil q$	$q \lceil p$
$p < q$	q	q
$p \geq q$	p	p

Table 3.9 Commutativity of the maximum function

Associativity. To prove that

$$p \lceil (q \lceil r) \equiv (p \lceil q) \lceil r$$

it is necessary to examine the six cases specified in the first column of

Table 3.10. For the first case, execution of Program 3.2 yields $q \lceil r \equiv r$,

Case	Associativity		Distributivity	
	$p \lceil (q \lceil r)$	$(p \lceil q) \lceil r$	$p \lfloor (q \lceil r)$	$(p \lfloor q) \lceil (p \lfloor r)$
$p \leq q \leq r$	r	r	p	p
$p \leq r \leq q$	q	q	p	p
$q \leq p \leq r$	r	r	p	p
$q \leq r \leq p$	p	p	r	r
$r \leq p \leq q$	q	q	p	p
$r \leq q \leq p$	p	p	q	q

Table 3.10 Derivation of some fundamental properties of $\lceil$ and $\lfloor$

and a second execution yields $p \lceil r \equiv r$; hence $p \lceil (q \lceil r) \equiv r$ as shown in column 2. Similarly, $(p \lceil q) \lceil r \equiv r$ as shown in column 3. Continuing for the remaining cases yields columns 2 and 3. Since they are identical, it follows that $p \lceil (q \lceil r) \equiv (p \lceil q) \lceil r$ for all values of p, q, and r; the function $\lceil$ is therefore associative.

Distributivity. Columns 4 and 5 of Table 3.10 show the values of $p \lfloor (q \lceil r)$ and $(p \lfloor q) \lceil (p \lfloor r)$, respectively. Since agreement is shown in all possible cases, it can be concluded that minimum distributes over maximum. The reader will find it instructive to verify that addition distributes over maximum. Only two cases need be distinguished, as shown in Table 3.11.

Case	$p + q \lceil r$	$(p + q) \lceil (p + r)$
$q < r$	$p + r$	$p + r$
$q \geq r$	$p + q$	$p + q$

Table 3.11 Distributivity of + over $\lceil$

The methods used in constructing Tables 3.9 and 3.10 can be used to derive the properties of the other dyadic functions defined thus far. The results are summarized in Table 3.12, in which a 1 indicates that the property applies, and a 0 indicates that it does not apply. In the table for distributivity, the entries in the *i*th row indicate which

Commutativity

+	−	×	÷	⌈	⌊	*
1	0	1	0	1	1	0

Associativity

+	−	×	÷	⌈	⌊	*
1	0	1	0	1	1	0

Distributivity

	+	−	×	÷	⌈	⌊	*
+	0	0	0	0	1	1	0
−	0	0	0	0	0	0	0
×	1	1	0	0	0	0	0
÷	0	0	0	0	0	0	0
⌈	0	0	0	0	1	1	0
⌊	0	0	0	0	1	1	0
*	0	0	0	0	0	0	0

Table 3.12 Fundamental properties of some dyadic functions

functions the *i*th function distributes over. Thus row 1 shows that + distributes over ⌈ and ⌊, and *column* 1 shows that + is distributed over only by ×.

(Do Exercises 3.16–3.17.)

Properties of vector functions. Since dyadic functions are extended to vectors component by component, the fundamental properties of any function F apply directly to any vector function of the form $\mathbf{x} F \mathbf{y}$. For

$$(\mathbf{x} F \mathbf{y})_i \equiv \mathbf{x}_i F \mathbf{y}_i \text{ and } (\mathbf{y} F \mathbf{x})_i \equiv \mathbf{y}_i F \mathbf{x}_i$$

and if F is commutative, then $\mathbf{x}_i F \mathbf{y}_i \equiv \mathbf{y}_i F \mathbf{x}_i$. Finally, $(\mathbf{x} F \mathbf{y})_i \equiv (\mathbf{y} F \mathbf{x})_i$ for all i, and hence

$$\mathbf{x} F \mathbf{y} \equiv \mathbf{y} F \mathbf{x}$$

A similar argument applies to other properties; hence if F is associative,

$$\mathbf{x} F (\mathbf{y} F \mathbf{z}) \equiv (\mathbf{x} F \mathbf{y}) F \mathbf{z}$$

and if F distributes over G,

$$\mathbf{x} F (\mathbf{y} G \mathbf{z}) \equiv (\mathbf{x} F \mathbf{y}) G (\mathbf{x} F \mathbf{z})$$

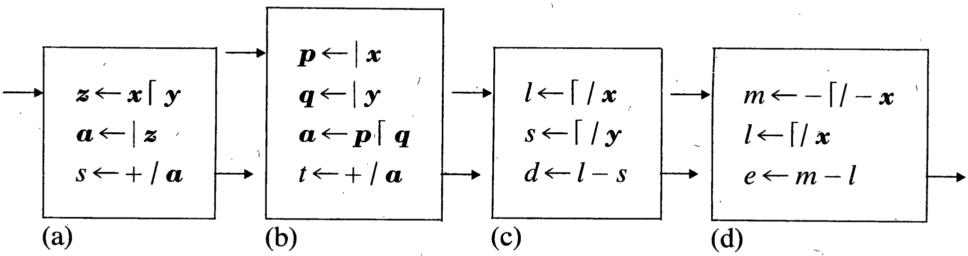
For example,

$$\mathbf{x} \times (\mathbf{y} + \mathbf{z}) \equiv (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z})$$

(Do Exercises 3.18–3.35.)

Exercises

3.1 Execute each of the following programs for the case $\boldsymbol{x} = 6, {}^{-}4, 16, 0, {}^{-}8$ and $\boldsymbol{y} = 4, 2, 7, {}^{-}6, {}^{-}10$.



(a)
```
z←x⌈y
a←|z
s←+/a
```

(b)
```
p←|x
q←|y
a←p⌈q
t←+/a
```

(c)
```
l←⌈/x
s←⌈/y
d←l−s
```

(d)
```
m←−⌈/−x
l←⌈/x
e←m−l
```

3.2 Using the maximum function ⌈, write a program for the solution of Exercise 2.24.

3.3 Using the maximum and minimum functions, write a one-statement program which determines whether all components of a vector $\boldsymbol{x}$ are equal; that is, determine d so that $d = 0$ if all components are equal and $d \neq 0$ otherwise.

3.4 Using the absolute-value and minimum functions, write a one-statement program to set $n = 0$ if all components of $\boldsymbol{x}$ are nonnegative and $n \neq 0$ otherwise.

3.5 Execute Program 3.6 for $m = 30$, $n = 42$, and then for $m = 42$, $n = 30$.

3.6 The integer vector $\boldsymbol{a}$ of dimension 2 represents the rational number $\div/\,\boldsymbol{a}$. Write a program to reduce $\boldsymbol{a}$ to lowest terms (see Program 3.5, but use vector functions as much as possible).

3.7 (a) Write a one-statement program to determine $\boldsymbol{r}$ as the vector of remainders on dividing the vector $\boldsymbol{x}$ by the integer n.

(b) Write a program to determine the greatest common divisor of the components of the vector $\boldsymbol{x}$.

3.8 (a) Write a program to determine $\boldsymbol{p}$ as the vector consisting of the prime numbers up to n.

(b) Execute your program for $n = 8$.

3.9 Write programs to determine, for any positive integer n,

(a) the vector $\boldsymbol{d}$ as the set of all distinct divisors of n

(b) the vector $\boldsymbol{q}$ as the set of all distinct prime divisors of n

3.10 Let $\boldsymbol{p}$ be a vector whose components are the first $\rho\,\boldsymbol{p}$ prime numbers arranged in ascending order; for example,

$(2,3,5,7)$ and $(2,3,5,7,11,13,17,19)$ are such prime vectors. Then the factors of any number n whose prime factors are contained in $\boldsymbol{p}$ can be displayed in an exponent vector $\boldsymbol{e}$ such that

$$n = \times/\boldsymbol{p} \ast \boldsymbol{e}$$

For example, if $n = 1176$ and $\boldsymbol{p} = (2,3,5,7,11)$, then $\boldsymbol{e} = (3,1,0,2,0)$, and $\boldsymbol{p} \ast \boldsymbol{e} = (8,3,1,49,1)$, and $\times/\boldsymbol{p} \ast \boldsymbol{e} = 1176$.

(a) Let $m = \times/\boldsymbol{p} \ast \boldsymbol{d}$, let $n = \times/\boldsymbol{p} \ast \boldsymbol{e}$, and let $r = \times/\boldsymbol{p} \ast \boldsymbol{g}$ be the greatest common divisor of m and n. Write a program to determine $\boldsymbol{g}$, using $\boldsymbol{d}$ and $\boldsymbol{e}$ as arguments.

(b) Execute your program for the case $\boldsymbol{d} = (2,2,1,1,1)$ and $\boldsymbol{e} = (3,1,0,2,0)$.

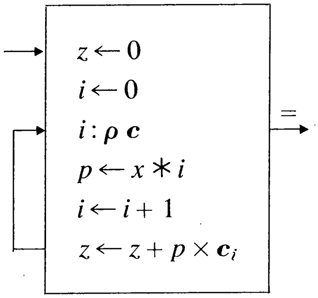
(c) Write a program (using $\boldsymbol{d}$ and $\boldsymbol{e}$ as arguments) to determine $\boldsymbol{l}$ such that $s = \times/\boldsymbol{p} \ast \boldsymbol{l}$, where s is the least common multiple of m and n.

(d) Execute the program of part (c) for the case given in part (b).

3.11 (a) Write a program which factors n with respect to the prime vector $\boldsymbol{p}$ (defined in Exercise 3.10) to yield the vector of exponents $\boldsymbol{e}$ such that $n = \times/\boldsymbol{p} \ast \boldsymbol{e}$. (Assume that the dimension of $\boldsymbol{p}$ is sufficiently large that $\boldsymbol{p}$ contains all prime factors of n.)

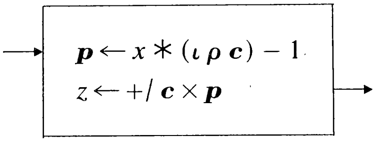
(b) Using the program of part (a), write a program to determine l as the least common multiple of the arguments a and b.

3.12 Execute the following programs for the case $x = 2$ and $\boldsymbol{c} = (1,3,0,2)$.



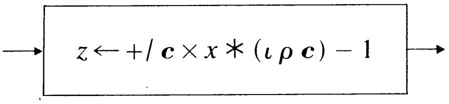
```
$z \leftarrow 0$
$i \leftarrow 0$
$i : \rho \boldsymbol{c}$   $=$
$p \leftarrow x \ast i$
$i \leftarrow i + 1$
$z \leftarrow z + p \times \boldsymbol{c}_i$
```

(a)



```
$\boldsymbol{p} \leftarrow x \ast (\iota \rho \boldsymbol{c}) - 1$
$z \leftarrow +/\boldsymbol{c} \times \boldsymbol{p}$
```

(b)



```
$z \leftarrow +/\boldsymbol{c} \times x \ast (\iota \rho \boldsymbol{c}) - 1$
```

(c)

3.13 State in words what well-known function all the programs of Exercise 3.12 represent.

3.14 Execute the following sequence of statements for the case $\mathbf{x} = 5, 6, {}^{-}2, 3, {}^{-}3$ and $\mathbf{y} = 2, 7, 1, 3, 4$, and for values of n from 1 to 5:

(a) $\mathbf{u} \leftarrow \mathbf{x} \leq \mathbf{y}$
(b) $p \leftarrow +/ (\mathbf{x} \leq \mathbf{y}) / \mathbf{x}$
(c) $\mathbf{v} \leftarrow 0 = (\mathbf{x} \leq \mathbf{y})$
(d) $q \leftarrow +/ (0 = \mathbf{x} \leq \mathbf{y}) / \mathbf{x}$
(e) $z \leftarrow p + q - +/ \mathbf{x}$
(f) $\mathbf{w} \leftarrow 2 \mid \iota 8$
(g) $\mathbf{d} \leftarrow (2 \mid \iota 8) / \iota 8$
(h) $e \leftarrow (+/\mathbf{d}) - (8 \div 2) * 2$
(i) $e \leftarrow (n * 2) - +/ (2 \mid \iota 2 \times n) / \iota 2 \times n$

3.15 (a) State in words what each statement of Exercise 3.14 does.

(b) Write a one-statement program which selects from $\mathbf{x}$ (to determine $\mathbf{z}$) those components of $\mathbf{x}$ which are divisible by the integer n.

(c) Write a one-statement program with integer arguments k and n which sums the integers 1, $1 + k$, $1 + 2 \times k$, $1 + 3 \times k, \ldots$ up to but not exceeding the positive integer n.

(d) Write one-statement programs for each of the following problems:

  (i) Determine t as the sum of all components of $\mathbf{s}$ which are divisible by 3.

  (ii) Determine i as the sum of all components of $\mathbf{s}$ which are integers.

3.16 To prove that a function F is commutative, it is necessary to show that $x\,F\,y = y\,F\,x$ for *all* possible values of x and y. To prove that F is *not* commutative, it is only necessary to exhibit *one* pair of values of x and y for which $(x\,F\,y) \neq (y\,F\,x)$.

(a) Prove that each of the functions designated as noncommutative by Table 3.12 is indeed noncommutative.

(b) Prove the nonassociativity of each function so designated in Table 3.12.

(c) Prove the nondistributivity of + over ÷, of × over ×, and of × over $\lceil$.

3.17 Construct a table of the form of Table 3.10 to prove

(a) that $\lceil$ distributes over itself (as indicated by Table 3.12)

(b) that $\lceil$ distributes over $\lfloor$

3.18 Choose some numerical value for each of the arguments and verify each of the following identities for the chosen values.

(a) $(+/\,\boldsymbol{c} \times x \ast (\iota\,\rho\,\boldsymbol{c}) - 1) + (+/\,\boldsymbol{d} \times x \ast (\iota\,\rho\,\boldsymbol{d}) - 1)$
$\equiv +/\,(\boldsymbol{c} + \boldsymbol{d}) \times x \ast (\iota\,\rho\,\boldsymbol{c}) - 1$, where $\rho\,\boldsymbol{c} = \rho\,\boldsymbol{d}$

(b) $(\boldsymbol{x}\,F\,\boldsymbol{y}), \boldsymbol{p}\,F\,\boldsymbol{q} \equiv (\boldsymbol{x}, \boldsymbol{p})\,F\,\boldsymbol{y}, \boldsymbol{q}$ (Choose any dyadic function for F.)

(c) $\boldsymbol{x} + (\boldsymbol{y} \lceil \boldsymbol{z}) \equiv (\boldsymbol{x} + \boldsymbol{y}) \lceil (\boldsymbol{x} + \boldsymbol{z})$

3.19 Prove each of the following identities, indicating clearly the properties of the functions used in each step of the proof. Where possible, use any of the first identities in the proofs of later ones.

(a) $+/\,(\boldsymbol{x} + \boldsymbol{y}) \equiv (+/\,\boldsymbol{x}) + (+/\,\boldsymbol{y})$

(b) $(\boldsymbol{c} + \boldsymbol{d}) \times \boldsymbol{q} \equiv (\boldsymbol{c} \times \boldsymbol{q}) + (\boldsymbol{d} \times \boldsymbol{q})$

(c) $+/\,(\boldsymbol{c} + \boldsymbol{d}) \times \boldsymbol{q} \equiv (+/\,\boldsymbol{c} \times \boldsymbol{q}) + (+/\,\boldsymbol{d} \times \boldsymbol{q})$

NOTE: The identity in part (c) is used in the first section of Chapter 4.

3.20 (a) Write a program to sort the vector $\boldsymbol{x}$ into ascending order; that is, to rearrange its components so that they occur in ascending order.

(b) Write a program to specify $\boldsymbol{r}$ as the vector of all *distinct* components of $\boldsymbol{y}$ arranged in any convenient order.

3.21 The vectors $\boldsymbol{s}$ and $\boldsymbol{d}$ of the same dimension together represent a hand of playing cards, $\boldsymbol{s}_i$ representing the *suit* of the *i*th card (with $\boldsymbol{s}_i$ equal to 1 for a club, 2 for a diamond, 3 for a heart, and 4 for a spade) and $\boldsymbol{d}_i$ representing its denomination (with 1 for an ace, 2 for a deuce, 3 for a three, and so on, up to 11 for a jack, 12 for a queen, and 13 for a king). For example, $\boldsymbol{s} = (3, 1, 4, 1, 4)$ and $\boldsymbol{d} = (8, 3, 12, 13, 6)$, represent the hand eight of hearts, three of clubs, queen of spades, king of clubs, and six of spades. Write programs to respecify $\boldsymbol{s}$ and $\boldsymbol{d}$ so that they represent the same hand but arranged in the following order:

(a) In decreasing order by suit, and within each suit in decreasing order by denomination. For example: $\boldsymbol{s} = (4, 4, 3, 1, 1)$ and $\boldsymbol{d} = (12, 6, 8, 13, 3)$.

(b) In decreasing order by denomination *within* increasing order by suit, that is, $\boldsymbol{s} = (1, 1, 3, 4, 4)$ and $\boldsymbol{d} = (13, 3, 8, 12, 6)$.

(c) In increasing order by denomination within increasing order by suit.

(d) In decreasing order by suit within decreasing order by denomination.

3.22 Devise a scheme for representing a hand of playing cards by a single vector $\boldsymbol{h}$ such that the ith component of $\boldsymbol{h}$ is a single number representing the ith card. Write programs to

(a) specify $\boldsymbol{h}$ as a function of the vectors $\boldsymbol{s}$ and $\boldsymbol{d}$ so that $\boldsymbol{h}$ represents the same hand in your scheme as $\boldsymbol{s}$ and $\boldsymbol{d}$ do in the scheme of Exercise 3.21

(b) respecify $\boldsymbol{h}$ (as a function of $\boldsymbol{h}$) according to each of the four arrangements required in Exercise 3.21

3.23 (a) Write a one-statement program for the problem of Exercise 2.12 (a).

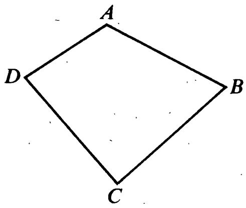
(b) Let $\boldsymbol{s}$ be a vector of dimension 3. Write a one-statement program to produce a result t such that $t = 1$ if the lengths $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, and $\boldsymbol{s}_3$ can form a triangle, and $t = 0$ otherwise.

3.24 Let the four vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ (each of dimension 2) be the plane coordinates of four points A, B, C, and D, no three of which are collinear.

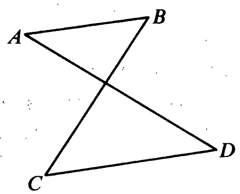
(a) Write a program to determine the equation of the line which passes through A and B, that is, determine the vector $\boldsymbol{e}$ of dimension 3 such that $+/\,\boldsymbol{e} \times \boldsymbol{a}, {}^{-}1 \equiv 0$ and $+/\,\boldsymbol{e} \times \boldsymbol{b}, {}^{-}1 \equiv 0$.

(b) Write a program (using part (a)) to set s to 1 if C and D are on the same side of the line through A and B and to 0 otherwise.

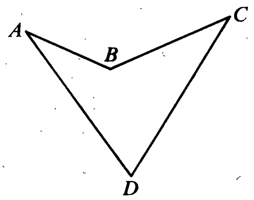
(c) Four vertices A, B, C, and D, given in a specified order, determine a figure of one of the following three types:



Type 1



Type 2



Type 3

Write a program to determine t as the type (1, 2, or 3) of the quadrilateral formed by the vertices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$.

- (d) Write a program which will (if possible) reorder the points to give a quadrilateral of Type 1 and to respecify t accordingly.
- (e) Assuming that the quadrilateral is of Type 1, write a program to determine the kind k, setting $k=4$ for a rhombus, 3 for a parallelogram, 2 for a trapezoid, and 1 otherwise.

3.25 Assuming that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are each in ascending order, write a program which merges the two into a single vector $\boldsymbol{s}$ arranged in ascending order.

3.26 A *perfect* number is one whose divisors (including itself and *one*) sum to twice the number. For example, 6 is a perfect number with divisors 1, 2, 3, 6, as is 28 with divisors 1, 2, 4, 7, 14, 28.

- (a) Write a program to determine $\boldsymbol{p}$ as the vector of all perfect numbers up to n.
- (b) If $+/2 \ast 0,\iota k$ is a prime number, then $m=(2 \ast k) \times +/2 \ast 0,\iota k$ is a perfect number, for it has the divisors

$$2 \ast 0,\iota k \text{ and } (+/2 \ast 0,\iota k) \times 2 \ast 0,\iota k$$

whose sum is clearly $(1++/2 \ast 0,\iota k) \times +/2 \ast 0,\iota k$

which is equal to $2 \times m$, since

$$1++/2 \ast 0,\iota k \equiv 2 \ast k+1 \equiv 2 \times 2 \ast k$$

All even perfect numbers are of this type; e.g., 6 and 28 are (for $k=1$ and 2 respectively). Write a program to generate all perfect numbers of this type for values of k up to some given limit l.

3.27 Write a program to determine the radius r and center $\boldsymbol{c}$ of the circle that passes through the points with plane coordinates $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$.

3.28 Write a program to determine s as the score for a bowling string where $\boldsymbol{p}$ is the vector of pins felled, that is, $\boldsymbol{p}_i$ are felled with the ith ball.

3.29 In the discussion of reduction over vectors of dimension zero, the identity element of a function F was defined as the value r such that $z\,F\,r \equiv z$ for all values of z. Such an element is more properly called a *right-identity* element, since one can also define a *left-identity* element as an element l which satisfies $l\,F\,z \equiv z$ for all values of z.

(a) Show that if F is commutative its left and right identities are equal.
(b) Make a three-column table showing all the dyadic functions listed in Table 3.12, together with their left- and right-identity elements (if they exist).
(c) Why is the use of the right-identity element appropriate to the present definition of the F-reduction of a vector?
(d) Exhibit a value for x that shows that $l|x$ cannot have a left identity.

3.30 If F is a commutative function which distributes over a function G, then

$$x\,F\,(y\,G\,z) \equiv (x\,F\,y)\;G\;(x\,F\,z) \tag{1}$$

and

$$(y\,G\,z)\;F\,x \equiv (y\,F\,x)\;G\;(z\,F\,x) \tag{2}$$

However, if F is noncommutative, then one of the preceding identities may hold while the other does not. For example, if F is *division* and G is *addition*, then

$$x \div (y + z) \equiv (x \div y) + (x \div z)$$

but

$$(y + z) \div x \equiv (y \div x) + (z \div x)$$

If identity (1) holds, then F is said to be *left-distributive* over G; if identity (2) holds, then F is said to be *right-distributive* over G.

(a) Prove that if F left-distributes over G and if F is commutative, then F right-distributes over G.
(b) Table 3.12 actually shows the left distributivity of the functions listed. Make a corresponding table showing right distributivity.

3.31 Certain identities that do not hold for all values of the arguments may hold for certain restricted values. For example, the identity

$$x \times (y \lceil z) \equiv (x \times y) \lceil (x \times z)$$

does not hold for all values of the arguments but does hold for all positive values. Modify Table 3.12 and the table required in Exercise 3.30 (b) to show the distributive properties if the arguments are restricted to positive integers.

3.32 Prove that
(a) $n|a + b \equiv n|(n|a) + n|b$

(b) $n|a\times b\equiv n|(n|a)\times n|b$

(c) $n|+/\boldsymbol{x}\equiv n|+/n|\boldsymbol{x}$

(d) $n|+/\boldsymbol{c}\times\boldsymbol{x}\equiv n|+/(n|\boldsymbol{c})\times n|\boldsymbol{x}$

(e) $9|10\ast k\equiv 1$ for any nonnegative integer k

3.33 Let $z=\ast/(m\ast 1\div m)\times 0\neq\iota n$.

(a) Compute z for $m=2$ and for values of n from 0 to 10.

(b) Compute z for $m=3$ and for values of n from 0 to 10.

(c) The result of part (a) clearly approaches 2; the result of part (b) would approach 3 for larger values of n. Prove the theorem suggested by parts (a) and (b).

3.34 (a) Write and execute a one-statement program to determine the value of $n\ast 1\div n$ for integer values of n from 1 to 10.

(b) From the results of part (a) it is clear that the function $n\ast 1\div n$ has a maximum value somewhere between $n=2$ and $n=4$. The value of n for which the maximum occurs will be called e. Write a one-statement program which evaluates the function for eleven equally spaced points from 2 to 4 to get a closer approximation for e.

(c) From the execution of part (b), select an even shorter interval which contains the maximum point e and again evaluate the function at equally spaced points.

(d) The process illustrated by the preceding parts can be continued to determine the position of the maximum e to any desired accuracy. Write a program to perform the entire process so as to determine e to two decimal places.

(e) The precise value of e can be shown (see Exercise 8.34) to be given by $e=+/1\div!0,\iota k$ for k sufficiently large. Evaluate this expression and compare with the result of part (d).

3.35 Consider the dyadic functions $\lceil,\lfloor,<,\leq,=,\geq,>,\neq$, where the arguments are logical variables, that is, their values are restricted to 0 and 1.

(a) Make a table showing the values of each of the functions for each of the four possible values of the arguments.

(b) Determine which of the functions are commutative.

(c) Determine which of the functions are associative.

(d) Make a table like Table 3.12 to show the distributive properties of this set of functions.

(e) Use a few sample values for the vector $\boldsymbol{x}$ to show that $\neq/\boldsymbol{x}\equiv 2|+/\boldsymbol{x}$.

Chapter Four

The Polynomial Function

Introduction

A function such as

$$4 + (6 \times x) + (3 \times x^2) + (5 \times x^4)$$

is commonly called a *polynomial in x* and is clearly a function of the argument x only. A function of the form

$$\boldsymbol{c}_1 + (\boldsymbol{c}_2 \times x) + (\boldsymbol{c}_3 \times x^2) + (\boldsymbol{c}_4 \times x^3) + (\boldsymbol{c}_5 \times x^4)$$

is also a polynomial in x, but is a function of both x and the vector of *coefficients* $\boldsymbol{c} = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_5)$; it will be denoted† by

$$\boldsymbol{c} \, \Pi \, x$$

The numerical example given at the outset would therefore be represented as

$$(4, 6, 3, 0, 5) \, \Pi \, x$$

The polynomial function is defined by any one of the three equivalent programs of Exercise 3.12, repeated here as Program 4.1.

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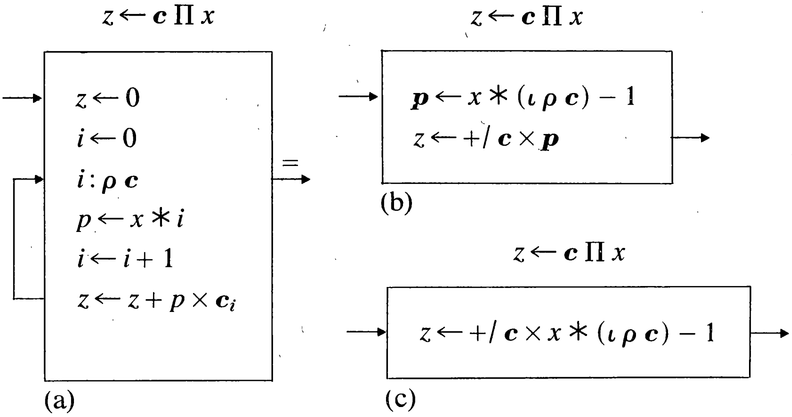
†Π is the Greek capital letter corresponding to the Roman letter P; it is chosen here to suggest the initial letter of *polynomial.*



```
z←c Π x
z←0
i←0
i:ρ c
p←x * i
i←i+1
z←z+p×c_i
```

(a)

```
z←c Π x
p←x * (ι ρ c) − 1
z←+/ c×p
```

(b)

```
z←c Π x
z←+/ c×x * (ι ρ c) − 1
```

(c)

Program 4.1 The polynomial function Π

The polynomial function owes its importance to four main factors:

1) Its evaluation for any given value of x requires only multiplication and addition.

2) It can be used to approximate any of the elementary functions as closely as desired.

3) It includes several functions of great utility: the quadratic function, which describes the parabola; the linear function, which describes the straight line; and the constant function.

4) Its properties, such as its slope (to be defined in Chapter 5) and the locations of its zeros†, are easily analyzed.

Because of its general importance, and because it will be used in analyzing the elementary functions introduced in later chapters, the polynomial function will be treated rather thoroughly. This treatment will include the addition and multiplication of polynomials, an efficient method of evaluation, the expansion of the function $(x+a) * n$ (that is, its expression as an equivalent polynomial $\boldsymbol{c}\,\Pi\, x$), and the approximation of other functions by means of polynomials.

Efficient Evaluation of Polynomials

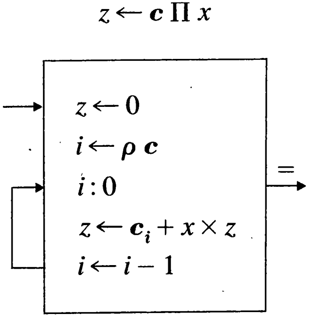
An alternative program for evaluating the polynomial $\boldsymbol{c}\,\Pi\, x$ can be derived by factoring as follows:

---

†A *zero* of a function of one argument is any value of its argument for which the value of the function is zero. For example, the zeros of the polynomial function $x^2 + x - 6$ are 2 and $^-3$.

$$
\begin{aligned}
\boldsymbol{c}\,\Pi\, x &\equiv \boldsymbol{c}_1 + (\boldsymbol{c}_2 \times x) + (\boldsymbol{c}_3 \times x^2) + (\boldsymbol{c}_4 \times x^3) + \ldots + (\boldsymbol{c}_{\rho \boldsymbol{c}} \times x^{(\rho \boldsymbol{c})-1}) \\
&\equiv \boldsymbol{c}_1 + x \times (\boldsymbol{c}_2 + (\boldsymbol{c}_3 \times x) + (\boldsymbol{c}_4 \times x^2) + \ldots + (\boldsymbol{c}_{\rho \boldsymbol{c}} \times x^{(\rho \boldsymbol{c})-2})) \\
&\equiv \boldsymbol{c}_1 + x \times (\boldsymbol{c}_2 + x \times (\boldsymbol{c}_3 + (\boldsymbol{c}_4 \times x) + \ldots + (\boldsymbol{c}_{\rho \boldsymbol{c}} \times x^{(\rho \boldsymbol{c})-3}))) \\
&\equiv \boldsymbol{c}_1 + x \times (\boldsymbol{c}_2 + x \times (\boldsymbol{c}_3 + x \times (\boldsymbol{c}_4 + \ldots (\boldsymbol{c}_{(\rho \boldsymbol{c})-1} + x \times (\boldsymbol{c}_{(\rho \boldsymbol{c})}) \ldots) \\
&\equiv \boldsymbol{c}_1 + x \times \boldsymbol{c}_2 + x \times \boldsymbol{c}_3 + x \times \boldsymbol{c}_4 + \ldots + x \times \boldsymbol{c}_{\rho \boldsymbol{c}}
\end{aligned}
$$

Program 4.2 describes this process, as can be verified by executing it for a general value of x so as to reconstruct the foregoing expression. For actual calculation, Program 4.2 is preferable to Programs 4.1 (a) through (c), since it requires far fewer multiplications than the other programs. Program 4.2 is therefore said to provide an *efficient* method for the evaluation of polynomials.



Program 4.2 Efficient evaluation of a polynomial

(Do Exercises 4.1–4.3.)

Degree of a Polynomial

The *degree* of a polynomial is the value of the largest exponent of x occurring in it. If the last component of $\boldsymbol{c}$ is nonzero, then the degree of $\boldsymbol{c}\,\Pi\, x$ is equal to $(\rho\, \boldsymbol{c}) - 1$.

The last component of $\boldsymbol{c}$ is normally assumed to be nonzero for the following reason. If Program 4.1 (c) is used to evaluate $\boldsymbol{c}\,\Pi\, x$ for the cases $\boldsymbol{c} = (4, 6, 3, 5)$ and $\boldsymbol{c} = (4, 6, 3, 5, 0, 0, 0)$, it will be clear that extending a coefficient vector by catenating zero components to the right makes no change in the polynomial it defines. More precisely,

$$(\boldsymbol{c}, 0 = \iota\, n)\, \Pi\, x \equiv \boldsymbol{c}\,\Pi\, x$$

for all values of x and n.

Addition of Polynomials

The function $(\boldsymbol{c}\,\Pi\, x) + \boldsymbol{d}\,\Pi\, x$ is called the *sum* of the polynomials $\boldsymbol{c}\,\Pi\, x$ and $\boldsymbol{d}\,\Pi\, x$. The sum of two polynomials is itself a polynomial $\boldsymbol{p}\,\Pi\, x$. In particular, if $\boldsymbol{c}$ and $\boldsymbol{d}$ have equal dimensions, then $\boldsymbol{p} \equiv \boldsymbol{c} + \boldsymbol{d}$. For if $\boldsymbol{q} \equiv x * (\iota\,\rho\,\boldsymbol{c}) - 1$, then, using the identity derived in Exercise 3.19,

$$\boldsymbol{p}\,\Pi\, x \equiv +/\,(\boldsymbol{c}+\boldsymbol{d}) \times \boldsymbol{q} \equiv (+/\,\boldsymbol{c}\times\boldsymbol{q}) + (+/\,\boldsymbol{d}\times\boldsymbol{q}) \equiv (\boldsymbol{c}\,\Pi\, x) + \boldsymbol{d}\,\Pi\, x$$

If $(\rho\,\boldsymbol{c}) > \rho\,\boldsymbol{d}$, then $(\boldsymbol{c}+\boldsymbol{d})$ is not defined, but the polynomials $\boldsymbol{c}\,\Pi\, x$ and $\boldsymbol{d}\,\Pi\, x$ can be added as follows:

$$(\boldsymbol{c}+\boldsymbol{d}\,,0=\iota\,(\rho\,\boldsymbol{c}) - \rho\,\boldsymbol{d})\,\Pi\, x \equiv (\boldsymbol{c}\,\Pi\, x) + (\boldsymbol{d}\,\Pi\, x)$$

for, as previously remarked,

$$(\boldsymbol{d}\,,0=\iota\, n)\,\Pi\, x \equiv \boldsymbol{d}\,\Pi\, x$$

A general expression for vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ of arbitrary dimension can be obtained by appending $(\rho\,\boldsymbol{c}) - \rho\,\boldsymbol{d}$ zeros to $\boldsymbol{d}$, and $(\rho\,\boldsymbol{d}) - \rho\,\boldsymbol{c}$ zeros to $\boldsymbol{c}$, except that the dimension of the vector of zeros may be negative in one of the cases, in which event a zero dimension is desired. Finally,

$$((\boldsymbol{c}\,,0=\iota\,0\lceil\,(\rho\,\boldsymbol{d}) - \rho\,\boldsymbol{c}) + \boldsymbol{d}\,,0=\iota\,0\lceil\,(\rho\,\boldsymbol{c}) - \rho\,\boldsymbol{d})\,\Pi\, x \equiv (\boldsymbol{c}\,\Pi\, x) + \boldsymbol{d}\,\Pi\, x$$

For example, if $\boldsymbol{c} = (3\,,1\,,5)$ and $\boldsymbol{d} = (2\,,{}^{-}4\,,0\,,3\,,2)$, then the coefficient vector on the left becomes

$$\begin{aligned}&(3\,,1\,,5\,,0=\iota\,0\lceil\,2) + (2\,,{}^{-}4\,,0\,,3\,,2\,,0=\iota\,0\lceil\,{}^{-}2)\\ &\equiv (3\,,1\,,5\,,0\,,0) + (2\,,{}^{-}4\,,0\,,3\,,2) \equiv (5\,,{}^{-}3\,,5\,,3\,,2)\end{aligned}$$

(Do Exercises 4.4–4.6.)

Multiplication of Polynomials

If $\boldsymbol{p}\,\Pi\, x \equiv (\boldsymbol{c}\,\Pi\, x) \times (\boldsymbol{d}\,\Pi\, x)$, then $\boldsymbol{p}\,\Pi\, x$ is called the *product* of polynomials $\boldsymbol{c}\,\Pi\, x$ and $\boldsymbol{d}\,\Pi\, x$. It is clear that the coefficient vector $\boldsymbol{p}$ is a function of $\boldsymbol{c}$ and $\boldsymbol{d}$.

Since $\boldsymbol{c}\,\Pi\, x$ is a sum of terms of the form $\boldsymbol{c}_i \times x^{i-1}$, and $\boldsymbol{d}\,\Pi\, x$ is a sum of terms of the form $\boldsymbol{d}_j \times x^{j-1}$, their product consists of the sum of all terms of the form

$$\boldsymbol{c}_i \times x^{i-1} \times \boldsymbol{d}_j \times x^{j-1} \equiv (\boldsymbol{c}_i \times \boldsymbol{d}_j) \times x^{i+j-2}$$

The values of the $((\rho\,\boldsymbol{c}) \times \rho\,\boldsymbol{d})$ coefficients $\boldsymbol{c}_i \times \boldsymbol{d}_j$ can be displayed in a rectangular array as shown in Table 4.3 for the case $\boldsymbol{c} = (3\,,1\,,2)$ and $\boldsymbol{d} = (4\,,0\,,1\,,3)$. The entry in the ith row and jth column is the product $\boldsymbol{c}_i \times \boldsymbol{d}_j$. It is obvious that all entries in the kth diagonal oval (counting from upper left to lower right) share the factor x^{k-1}; hence the coefficients $\boldsymbol{p}$ of the product polynomial are obtained by summing the entries in the successive ovals, as shown in Table 4.3.

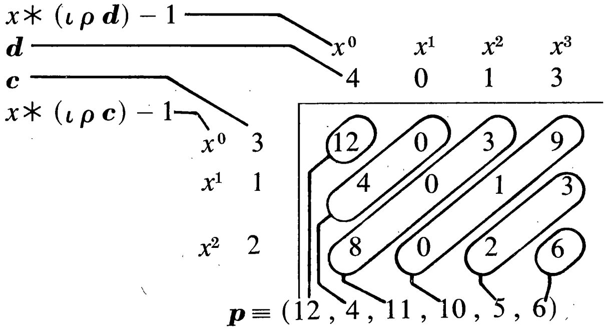


Table 4.3 Product of polynomials $\boldsymbol{p}\,\Pi\, x \equiv (\boldsymbol{c}\,\Pi\, x) \times (\boldsymbol{d}\,\Pi\, x)$

Two special cases of polynomial multiplication merit mention:

$$a \times (\boldsymbol{c}\,\Pi\, x) \equiv (a \times \boldsymbol{c})\,\Pi\, x$$

and

$$(x* n) \times (\boldsymbol{c}\,\Pi\, x) \equiv ((0 = \iota\, n), \boldsymbol{c})\,\Pi\, x$$

Both of these identities can be derived from the scheme of Table 4.3, since $a \equiv a\,\Pi\, x$ and $x * n \equiv ((0 = \iota\, n)\,, 1)\,\Pi\, x$, as the reader should verify.

(Do Exercises 4.7–4.11.)

Synthetic Division

If $\boldsymbol{n}\,\Pi\, x$ and $\boldsymbol{d}\,\Pi\, x$ are any two polynomials, then it is possible to find a *quotient* polynomial $\boldsymbol{q}\,\Pi\, x$ and a *remainder* polynomial $\boldsymbol{r}\,\Pi\, x$ such that $(\rho\, \boldsymbol{r}) < \rho\, \boldsymbol{d}$ and

$$\boldsymbol{n}\,\Pi\, x \equiv ((\boldsymbol{d}\,\Pi\, x) \times \boldsymbol{q}\,\Pi\, x) + \boldsymbol{r}\,\Pi\, x$$

This is analogous to the division of an integer n by an integer divisor d to obtain an *integer quotient* q and an *integer remainder* r which is less than d.

For example, if

$$\boldsymbol{n} = 3\,, 7\,, 0\,, 7\,, {}^{-}10\,, 8 \text{ and } \boldsymbol{d} = 3\,, 1\,, 4$$

then

$$\boldsymbol{q} = 2\,, 1\,, {}^{-}3\,, 2 \text{ and } \boldsymbol{r} = {}^{-}3\,, 2$$

as can be verified by multiplying $\boldsymbol{d}\,\Pi\, x$ and $\boldsymbol{q}\,\Pi\, x$ and then adding $\boldsymbol{r}\,\Pi\, x$ as shown in Figure 4.4 (d).

In the present example, $\boldsymbol{q}$ and $\boldsymbol{r}$ can be determined from $\boldsymbol{n}$ and $\boldsymbol{d}$ as follows. First, from $\boldsymbol{n}\,\Pi\, x$ subtract the polynomial

$$
\begin{aligned}
&((0\,,0\,,0\,,\boldsymbol{n}_{\rho n} \div \boldsymbol{d}_{\rho d})\,\Pi\, x) \times (\boldsymbol{d}\,\Pi\, x)\\
&\equiv ((0\,,0\,,0\,,2)\,\Pi\, x) \times (3\,,1\,,4)\,\Pi\, x\\
&\equiv (0\,,0\,,0\,,6\,,2\,,8)\,\Pi\, x
\end{aligned}
$$

This yields the remainder

$$(3\,,7\,,0\,,1\,,{}^{-}12\,,0)\,\Pi\, x$$

Since the coefficient $\boldsymbol{n}_{\rho n} \div \boldsymbol{d}_{\rho d}$ was chosen to make the final coefficient zero, the remainder is of degree $(\rho\, \boldsymbol{n}) - 2$ and can be written as

$$(3\,,7\,,0\,,1\,,{}^{-}12)\,\Pi\, x$$

The process can be repeated by subtracting from the remainder another multiple of $\boldsymbol{d}\,\Pi\, x$, so chosen as to further reduce the degree of the new remainder, and can be continued until the degree of the final remainder is less than the degree of $\boldsymbol{d}\,\Pi\, x$.

The entire process is called *synthetic division* and is described by the program of Figure 4.4. The program is accompanied by an execution (Figure 4.4 (b)) for the case

$$\boldsymbol{n} = 3\,,7\,,0\,,7\,,{}^{-}10\,,8 \text{ and } \boldsymbol{d} = 3\,,1\,,4$$

Figure 4.4 (c) shows a convenient arrangement for the manual execution of synthetic division,† and Figure 4.4 (d) shows the verification of the result.

---

†Readers may be familiar with an arrangement in which the division process begins at the left rather than the right. The difference results from the present choice of order for the terms of the polynomial (that is, in ascending powers of the argument x). This is opposite to the order often used and has been chosen here to facilitate the treatment of polynomials of unlimited degree, which will be used extensively in later chapters.

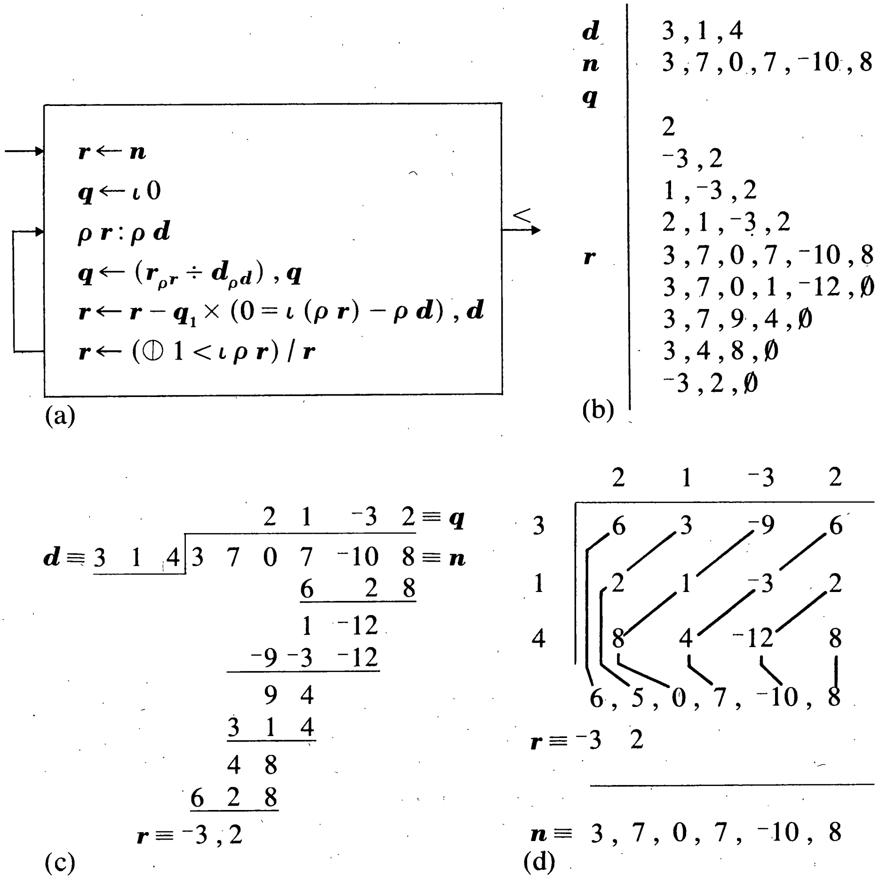


Figure 4.4 Synthetic division

If $\boldsymbol{d}$ is the vector $((-a), 1)$, then the remainder $\boldsymbol{r}$ is a single quantity (that is, $\rho\,\boldsymbol{r} \equiv 1$) and is the value of $\boldsymbol{n} \,\Pi\, a$. For

$$\begin{aligned} \boldsymbol{n}\,\Pi\, x &\equiv ((\boldsymbol{q}\,\Pi\, x) \times \boldsymbol{d}\,\Pi\, x) + \boldsymbol{r}\,\Pi\, x \\ &\equiv ((\boldsymbol{q}\,\Pi\, x) \times (-a) + x) + \boldsymbol{r}_1 \end{aligned}$$

and therefore

$$\boldsymbol{n}\,\Pi\, a \equiv ((\boldsymbol{q}\,\Pi\, a) \times (-a) + a) + \boldsymbol{r}_1 \equiv \boldsymbol{r}_1$$

Synthetic division by $(x - a)$ therefore serves to evaluate the polynomial for the value a of the argument x, and the computational procedure is in fact equivalent to the method of Program 4.2.

(Do Exercises 4.12–4.15.)

The Binomial Theorem

If n is a positive integer, the function $(a+x) \ast n$ can be expressed as a polynomial in x; for example,

$$\begin{aligned}
&(a+x) \ast 0 \equiv 1\\
&(a+x) \ast 1 \equiv a+x\\
&(a+x) \ast 2 \equiv a^2 + (2 \times a \times x) + x^2\\
&(a+x) \ast 3 \equiv a^3 + (3 \times a^2 \times x) + (3 \times a \times x^2) + x^3
\end{aligned}$$

The binomial theorem provides a simple general scheme for determining the coefficients in such a *polynomial expansion* of $(a+x) \ast n$. Attention will first be restricted to the simple case for $a = 1$ (that is, $(1+x) \ast n$), since the more general case can be derived from it.

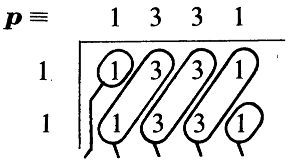
Expansion of $(1+x) \ast n$. If $\boldsymbol{p}$ is the vector of coefficients of the polynomial expansion of $(1+x) \ast n$ (that is, $(1+x) \ast n \equiv \boldsymbol{p} \,\Pi\, x$), and if $\boldsymbol{q}$ is the corresponding vector for $(1+x) \ast n+1$, then since $(1+x) \ast n+1 \equiv (1+x) \times (1+x) \ast n$, it follows that

$$\begin{aligned}
\boldsymbol{q} \,\Pi\, x &\equiv (1+x) \times \boldsymbol{p} \,\Pi\, x\\
&\equiv ((1\,,1) \,\Pi\, x) \times \boldsymbol{p} \,\Pi\, x
\end{aligned}$$

The last expression is a product of two polynomials and so can be evaluated by the method of Table 4.3. For example, if $n = 3$, then

$$(1+x) \ast 3 \equiv \boldsymbol{p} \,\Pi\, x \equiv (1\,,3\,,3\,,1) \,\Pi\, x$$

Similarly, the coefficients $\boldsymbol{q}$ of the expansion of $(1+x) \ast 4$ can be obtained as follows:



$\boldsymbol{q} \equiv (1\,,4\,,6\,,4\,,1)$

From this example it is clear that, in general, $\boldsymbol{q}$ is determined from $\boldsymbol{p}$ as follows:

$$\boldsymbol{q} \leftarrow (0\,,\boldsymbol{p}) + (\boldsymbol{p}\,,0)$$

This rule can be applied to generate the coefficients for successive powers of $(1+x)$ as shown in the successive rows of Table 4.5; the table is called *Pascal's triangle.*

n											
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Table 4.5 Pascal's triangle

The coefficient vector of the expansion of $(1+x)^n$, which appears as the $(n+1)$th row of Pascal's triangle, is clearly a monadic function of n; it will be denoted† by β. Thus $\beta\, 0 \equiv 1$; $\beta\, 1 \equiv (1, 1)$; $\beta\, 2 \equiv (1, 2, 1)$; and so forth. In general, then,

$$(1+x) \ast n \equiv (\beta\, n)\, \Pi\, x$$

EXAMPLE: Computation of compound interest furnishes an example of the use of the expansion of $(1+x) \ast n$. If a certain amount of capital c is invested and left to accumulate at an interest rate of x percent compounded annually, then the amounts accumulated at the end of the first, second, and third years are $c \times (1+x)$; $c \times (1+x) \ast 2$; and $c \times (1+x) \ast 3$. In general, the amount accumulated at the end of the nth year is $c \times (1+x) \ast n$. The value of $(1+x) \ast n$ can be conveniently computed from its expansion. For example, if $n = 7$ years, and $x = .03$ (that is, 3 percent), then from Table 4.5, $\beta\, 7 \equiv (1, 7, 21, 35, 35, 21, 7, 1)$, and

$$\begin{aligned}(1+.03) \ast 7 &\equiv 1 + (7 \times .03) + (21 \times .03^2) + (35 \times .03^3) + \ldots\\ &\equiv 1 + .21 + .0189 + .000945 + \ldots\\ &\equiv 1.229845 + \ldots\end{aligned}$$

This result is correct to four decimal places.

Expansion of $(a+x) \ast n$. Since

$$(x \times y) \ast n \equiv (x \ast n) \times (y \ast n)$$

---

†The Greek letter β corresponds to the Roman letter b, and is used here because b is the initial letter of *binomial.* It is spelled *beta* and pronounced *bayta.*

and since

$$(a+x)\ast n \equiv (a\times(1+x\div a))\ast n$$

then

$$\begin{aligned}(a+x)\ast n &\equiv (a\ast n)\times(1+x\div a)\ast n\\ &\equiv (a\ast n)\times(\beta\, n)\,\Pi\, x\div a\\ &\equiv (a\ast n)\times +/\ (\beta\, n)\times(x\div a)\ast 0\,,\iota\, n\\ &\equiv (a\ast n)\times +/\ (\beta\, n)\times((1\div a)\ast 0\,,\iota\, n)\times x\ast 0\,,\iota\, n\\ &\equiv (a\ast n)\times +/\ (\beta\, n)\times(a\ast -0\,,\iota\, n)\times x\ast 0\,,\iota\, n\\ &\equiv +/\ (\beta\, n)\times(a\ast n)\times(a\ast -0\,,\iota\, n)\times x\ast 0\,,\iota\, n\\ &\equiv +/\ (\beta\, n)\times(a\ast n-0\,,\iota\, n)\times x\ast 0\,,\iota\, n\end{aligned}$$

But $n-0\,,\iota\, n\equiv \text{⌽}\, 0\,,\iota\, n$. For example, if $n\equiv 4$, then $n-0\,,\iota\, n\equiv$ $4-(0\,,1\,,2\,,3\,,4)\equiv(4\,,3\,,2\,,1\,,0)\equiv \text{⌽}\, 0\,,\iota\, n$.
Therefore

$$\begin{aligned}(a+x)\ast n &\equiv +/\ (\beta\, n)\times(a\ast \text{⌽}\, 0\,,\iota\, n)\times x\ast 0\,,\iota\, n\\ &\equiv +/\ (\beta\, n)\times(\text{⌽}\, a\ast 0\,,\iota\, n)\times x\ast 0\,,\iota\, n\end{aligned}$$

or

$$(a+x)\ast n\equiv((\beta\, n)\times \text{⌽}\, a\ast 0\,,\iota\, n)\,\Pi\, x \qquad (4.1\text{ a})$$

Since $(a+x)\ast n\equiv(x+a)\ast n$, it is clear that the roles of x and a can be interchanged, and hence $(a+x)\ast n$ can be written as a polynomial in a with coefficients that depend on x. Thus

$$(a+x)\ast n\equiv((\beta\, n)\times \text{⌽}\, x\ast 0\,,\iota\, n)\,\Pi\, a \qquad (4.1\text{ b})$$

For example:

$$\begin{aligned}(a+x)\ast 4 &\equiv((1\,,4\,,6\,,4\,,1)\times(a^4\,,a^3\,,a^2\,,a^1\,,a^0))\,\Pi\, x\\ &\equiv((1\,,4\,,6\,,4\,,1)\times(x^4\,,x^3\,,x^2\,,x^1\,,x^0))\,\Pi\, a\end{aligned}$$

Equations 4.1 (a) and (b) are two commonly used forms of the binomial theorem.

EXAMPLE: The polynomial $(2\,,1\,,3\,,5\,,1)\,\Pi\, x+2$ is clearly equivalent to some polynomial $\boldsymbol{p}\,\Pi\, x$. The value of $\boldsymbol{p}$ can be determined by applying the binomial theorem to each term of the given polynomial as follows:

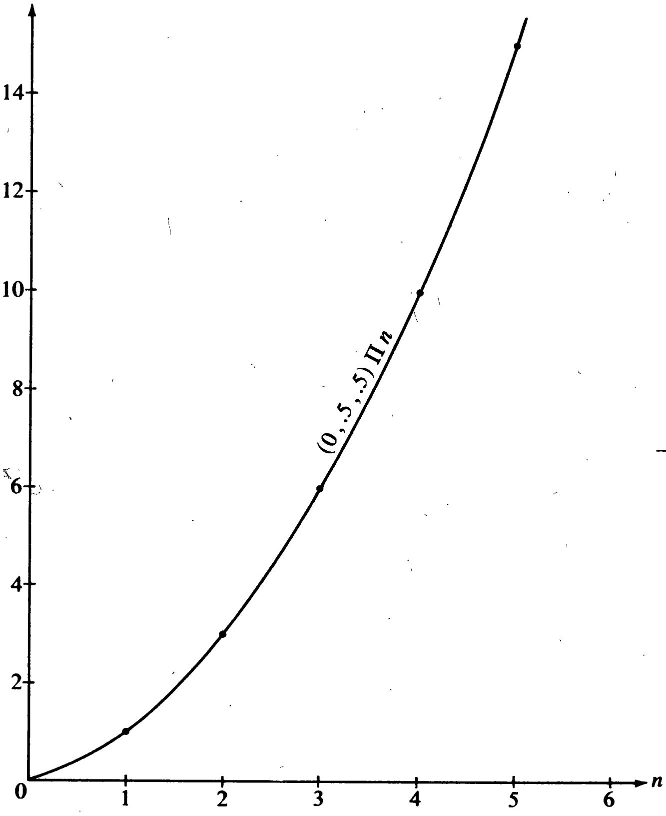
$$\begin{array}{lr}2\times 1\times 1 & =2\\ 1\times(1\,,1)\times(2\,,1) & =2\,,1\\ 3\times(1\,,2\,,1)\times(4\,,2\,,1) & =12\,,12\,,3\\ 5\times(1\,,3\,,3\,,1)\times(8\,,4\,,2\,,1) & =40\,,60\,,30\,,5\\ 1\times(1\,,4\,,6\,,4\,,1)\times(16\,,8\,,4\,,2\,,1) & =\underline{16\,,32\,,24\,,8\,,1}\\ & \boldsymbol{p}=72\,,105\,,57\,,13\,,1\end{array}$$

(Do Exercises 4.16–4.21.)

Approximation by Polynomials

If F is any monadic function and if $\boldsymbol{p}$ and $\boldsymbol{q}$ are vectors such that $\boldsymbol{q} \equiv F\,\boldsymbol{p}$ (that is, $\boldsymbol{q}_i \equiv F\,\boldsymbol{p}_i$ for $i \equiv 1, 2, \ldots, \rho\,\boldsymbol{p}$), then it is possible to determine a vector $\boldsymbol{c}$ of the same dimension as $\boldsymbol{p}$ such that $\boldsymbol{c}\,\Pi\,\boldsymbol{p}_i \equiv F\,\boldsymbol{p}_i$ for $i \equiv 1, 2, \ldots, \rho\,\boldsymbol{p}$. In other words, it is possible to find a polynomial of degree $(\rho\,\boldsymbol{p}) - 1$ or less which fits the function F at the $\rho\,\boldsymbol{p}$ points $(\boldsymbol{p}_i\,, F\,\boldsymbol{p}_i)$. By choosing a large number of points (that is, by choosing a large value for $\rho\,\boldsymbol{p}$), it is possible to fit the function F closely.

Consider, for example, the function $F\,n \equiv +/\,\iota\,n$ which is defined for all positive integers n and is both tabulated and plotted in Figure 4.6. If the quadratic polynomial $\boldsymbol{c}\,\Pi\,n$ is required to fit the three points $(1\,,1)$, $(2\,,3)$, and $(3\,,6)$, then it is necessary that $\boldsymbol{c}\,\Pi\,1 \equiv 1$ and



n	$+/\,\iota\,n$
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45

Figure 4.6 Polynomial approximation to $+/\,\iota\,n$

$\boldsymbol{c} \Pi 2 \equiv 3$ and $\boldsymbol{c} \Pi 3 \equiv 6$. In other words (since $\rho\, \boldsymbol{c} \equiv 3$ for a quadratic):

$$\begin{aligned} \boldsymbol{c}_1 + (\boldsymbol{c}_2 \times 1) + (\boldsymbol{c}_3 \times 1^2) &\equiv 1 \\ \boldsymbol{c}_1 + (\boldsymbol{c}_2 \times 2) + (\boldsymbol{c}_3 \times 2^2) &\equiv 3 \\ \boldsymbol{c}_1 + (\boldsymbol{c}_2 \times 3) + (\boldsymbol{c}_3 \times 3^2) &\equiv 6 \end{aligned}$$

Since the foregoing are three linear equations in the three variables $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and $\boldsymbol{c}_3$, they can be solved to obtain the solution $\boldsymbol{c} = (0\,,.5\,,.5)$. Hence the polynomial

$$\boldsymbol{c} \Pi n \equiv 0 + \frac{n^1}{2} + \frac{n^2}{2} \equiv \frac{n \times n + 1}{2}$$

fits the function $+/\, \iota\, n$ at the chosen points.

This polynomial fits the given function for the tabulated values, as the reader may verify. Actually, the polynomial fits the function for *all* integer values of n, and this can be shown by a simple proof (Exercise 4.22).

As a second example, consider the function $z \leftarrow +/\, (\iota\, n) \ast 2$ (that is, the sum of the first n squares), tabulated and graphed in Figure 4.7. Fitting the first three points by a quadratic yields the following equations:

$$\begin{aligned} \boldsymbol{c}_1 + \quad \boldsymbol{c}_2 \quad + \quad \boldsymbol{c}_3 \quad &\equiv \ \ 1 \\ \boldsymbol{c}_1 + (2 \times \boldsymbol{c}_2) + (4 \times \boldsymbol{c}_3) &\equiv \ \ 5 \\ \boldsymbol{c}_1 + (3 \times \boldsymbol{c}_2) + (9 \times \boldsymbol{c}_3) &\equiv 14 \end{aligned}$$

Their solution is given by

$$\boldsymbol{c} = 2\,,{}^{-}3.5\,,2.5$$

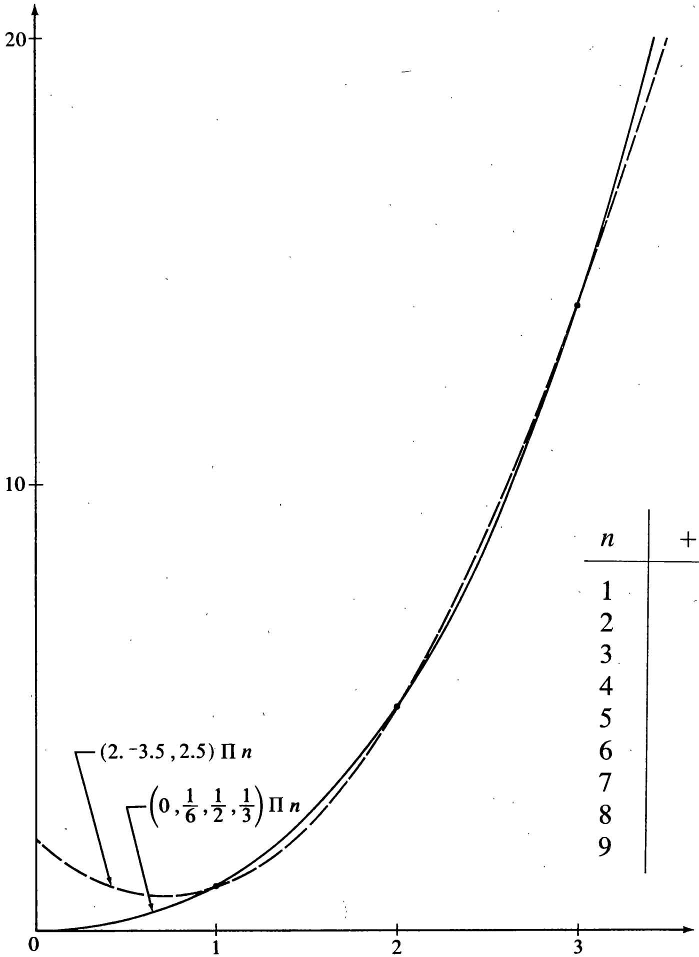
Although the polynomial $\boldsymbol{c} \Pi n$ does indeed fit the first three points, its graph shows it to be a very poor approximation for large values of n.

A better approximation can be obtained by using a polynomial of a higher degree. Choosing degree 4 and fitting to the first five points yields

$$\boldsymbol{c} \equiv 0\,,\frac{1}{6}\,,\frac{1}{2}\,,\frac{1}{3}\,,0$$

as can be verified by calculating the first five values of $\boldsymbol{c} \Pi n$. This polynomial fits the function exactly for all values of n. Moreover, since $\boldsymbol{c}_5 \equiv 0$, it is in fact of degree 3, and may be written as

$$\begin{aligned} +/\,(\iota\, n) \ast 2 &\equiv \left(0\,,\frac{1}{6}\,,\frac{1}{2}\,,\frac{1}{3}\right) \Pi n \\ &\equiv (n \times (n + .5) \times n + 1) \div 3 \end{aligned}$$



n	+/ (ι n) * 2
1	1
2	5
3	14
4	30
5	55
6	91
7	140
8	204
9	285

Figure 4.7 Polynomial approximations to +/ (ι *n*) * 2

From this example it appears that the use of an unnecessarily high degree for an approximating polynomial simply leads to zero coefficients for the higher-power terms of the polynomial; it therefore yields the same result as the use of a suitable lower degree but at the expense of more labor in calculating the coefficients. Finally, from the preceding *two* examples it may be surmised (correctly) that the sum of the *k*th powers of the first *n* integers can be fitted exactly by a polynomial of degree *k* + 1.

The functions of the foregoing examples were defined only for integral values of n. Nevertheless, the same method can be used to approximate a function that is defined everywhere; the function is simply fitted exactly for some selected values of the argument.

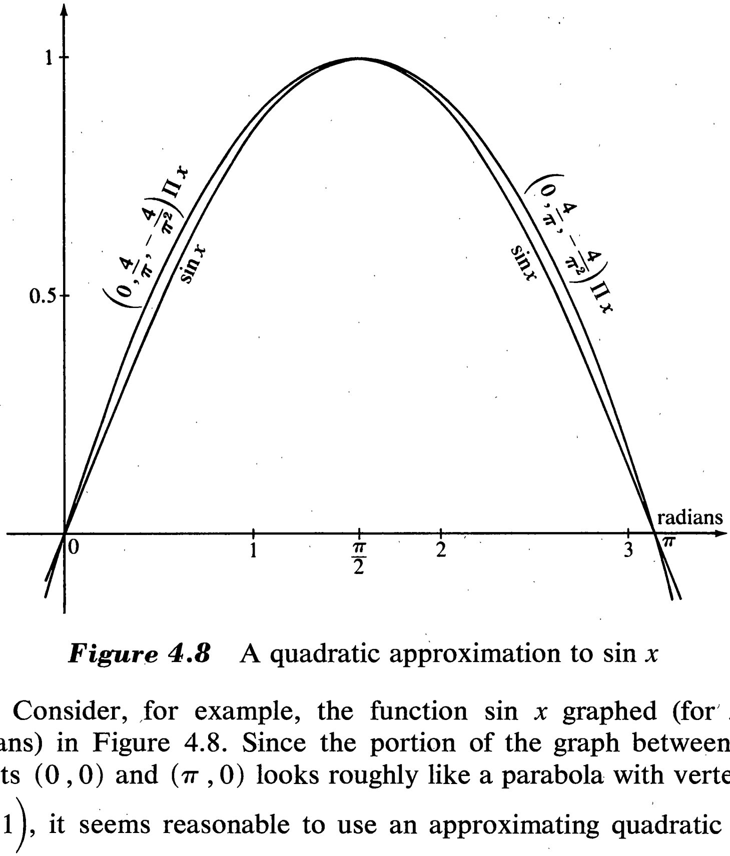


Figure 4.8 A quadratic approximation to sin x

Consider, for example, the function sin x graphed (for x in radians) in Figure 4.8. Since the portion of the graph between the points $(0, 0)$ and $(\pi, 0)$ looks roughly like a parabola with vertex at $\left(\frac{\pi}{2}, 1\right)$, it seems reasonable to use an approximating quadratic that passes through these three points. The coefficients must therefore satisfy the following equations:

$$\begin{aligned} \boldsymbol{c}_1 + (\boldsymbol{c}_2 \times 0) + (\boldsymbol{c}_3 \times 0) &\equiv 0 \\ \boldsymbol{c}_1 + \left(\boldsymbol{c}_2 \times \frac{\pi}{2}\right) + \left(\boldsymbol{c}_3 \times \frac{\pi^2}{4}\right) &\equiv 1 \\ \boldsymbol{c}_1 + (\boldsymbol{c}_2 \times \pi) + (\boldsymbol{c}_3 \times \pi^2) &\equiv 0 \end{aligned}$$

Therefore

$$\boldsymbol{c} \equiv \left(0, \frac{4}{\pi}, -\frac{4}{\pi^2}\right)$$

As shown by the curves of Figure 4.8, the quadratic function $\boldsymbol{c}\,\Pi\,x$ fits the function sin x reasonably well, but *only* from about $(0\,,0)$ to $(\pi\,,0)$. A better approximation could obviously be obtained by fitting a quadratic to points such as $(0.2\,,0.198)$, $\left(\frac{\pi}{2}\,,1\right)$, and $((\pi-0.2)\,,0.198)$. A better approximation could also be obtained by fitting to a larger number of points with a polynomial of higher degree.

(Do Exercises 4.22-4.26.)

Solution of linear equations. From the foregoing it is clear that any program for determining the coefficients of a polynomial to approximate a given function must contain within it a program to solve a set of linear equations. Although the reader is probably familiar with manual methods for the solution of sets of linear equations, it may be well to review the programming of such methods.

To facilitate this review it will be convenient to introduce notation for a two-dimensional array or *matrix*. A matrix will be denoted by a boldface uppercase letter, and a typical matrix $\boldsymbol{M}$ would appear as

$$\boldsymbol{M} \equiv \begin{array}{cccc} 3 & 7 & 2 & 8 \\ 4 & 8 & 16 & 2 \\ 2 & 1 & 3 & {}^{-}6 \end{array}$$

The ith row of a matrix $\boldsymbol{M}$ is a vector denoted by $\boldsymbol{M}^i$. The jth column of $\boldsymbol{M}$ is a vector denoted by $\boldsymbol{M}_j$. For example,

$$\boldsymbol{M}^1 \equiv (3\,,7\,,2\,,8) \text{ and } \boldsymbol{M}^2 \equiv (4\,,8\,,16\,,2)$$

while

$$\boldsymbol{M}_1 \equiv (3\,,4\,,2) \text{ and } \boldsymbol{M}_4 \equiv (8\,,2\,,{}^{-}6)$$

Moreover $\boldsymbol{M}^i_j$ denotes the component in the ith row and jth column. Thus $\boldsymbol{M}^1_1 \equiv 3$, $\boldsymbol{M}^1_3 \equiv 2$, and $\boldsymbol{M}^2_3 \equiv 16$.

Thus the matrix, like the vector, represents a family of variables. Unlike the vector, it represents a *two-dimensional* family, which requires two arguments (a row index and a column index) to identify an individual member of the family. A matrix can also be construed as a family of row vectors $\boldsymbol{M}^i$ or as a family of column vectors $\boldsymbol{M}_j$.

The *dimension* of a matrix $\boldsymbol{M}$ is the two-component vector $(m\,,n)$, where m is the number of rows and n the number of columns in the matrix $\boldsymbol{M}$. The dimension of $\boldsymbol{M}$ is denoted by $\rho\,\boldsymbol{M}$; in the present example, $\boldsymbol{M}$ has 3 rows and 4 columns, and therefore

$$\rho\,\boldsymbol{M} \equiv (3\,,4)$$

A matrix whose dimension is $(m\,,n)$ is often called an m-by-n matrix.

Since $\rho\,\boldsymbol{M}$ is a vector of dimension 2 it is consistent to define $\rho\,\boldsymbol{x}$ as a vector of dimension 1 rather than as a scalar and (since a scalar y can be construed as a zero-dimensional array) to define $\rho\,y$ as a vector of dimension 0.

An equation of the form

$$(3\times\boldsymbol{x}_1)+(6\times\boldsymbol{x}_2)+(2\times\boldsymbol{x}_3)\equiv 8$$

is called a *linear equation* in $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$. It can obviously be written as $(+/\,(3\,,6\,,2)\times\boldsymbol{x})\equiv 8$. More generally, $(+/\,\boldsymbol{c}\times\boldsymbol{x})\equiv b$ represents a linear equation with coefficients $\boldsymbol{c}$ and constant term b. The entire equation can be represented by the vector $\boldsymbol{d}=\boldsymbol{c}\,,b$, where a solution $\boldsymbol{x}$ must satisfy the relation $(+/\,\boldsymbol{d}\times\boldsymbol{x}\,,{}^{-}1)\equiv 0$. Moreover it is clear that for $a\neq 0$ the equation $a\times\boldsymbol{d}$ has the same solution as does $\boldsymbol{d}$. Finally, if $\boldsymbol{d}$ and $\boldsymbol{e}$ represent two equations that are satisfied by the same variables $\boldsymbol{x}$, the vector $\boldsymbol{d}+a\times\boldsymbol{e}$ has the same solution as do $\boldsymbol{d}$ and $\boldsymbol{e}$.

If $\boldsymbol{M}$ is any matrix, then each row vector $\boldsymbol{M}^i$ can be thought of as representing an equation in the variables $\boldsymbol{x}$, where $\rho\,\boldsymbol{x}\equiv(\rho\,\boldsymbol{M})_2-1$. The matrix then represents $(\rho\,\boldsymbol{M})_1$ equations in $(\rho\,\boldsymbol{M})_2-1$ variables, and if $(\rho\,\boldsymbol{M})_1\equiv(\rho\,\boldsymbol{M})_2-1$ there is normally one solution†, that is, one value of the vector $\boldsymbol{x}$ that satisfies all the equations

$$+/\,(\boldsymbol{M}^i\times\boldsymbol{x}\,,{}^{-}1)\equiv 0 \text{ for } i=1,2,\ldots,(\rho\,\boldsymbol{M})_1$$

For example, the matrix

$$\boldsymbol{M}\equiv\begin{matrix}2 & 6 & 4 & 2\\ 3 & 12 & 3 & 9\\ 2 & 1 & 4 & 2\end{matrix}$$

represents the set of linear equations

$$\begin{aligned}(2\times\boldsymbol{x}_1)+(6\times\boldsymbol{x}_2)+(4\times\boldsymbol{x}_3)-2&\equiv 0\\ (3\times\boldsymbol{x}_1)+(12\times\boldsymbol{x}_2)+(3\times\boldsymbol{x}_3)-9&\equiv 0\\ (2\times\boldsymbol{x}_1)+\boldsymbol{x}_2+(4\times\boldsymbol{x}_3)-2&\equiv 0\end{aligned}$$

Dividing $\boldsymbol{M}^1$ by $\boldsymbol{M}^1_1$ yields a new set of equations (which will again be denoted by $\boldsymbol{M}$):

$$\boldsymbol{M}\equiv\begin{matrix}1 & 3 & 2 & 1\\ 3 & 12 & 3 & 9\\ 2 & 1 & 4 & 2\end{matrix}$$

---

†There can also be an unlimited number of solutions or none. See, for example, Earle B. Miller and Robert M. Thrall, *College Algebra* (Ginn, 1950), Sec. 125.

These have the same solution as the original set. Furthermore, subtracting $\boldsymbol{M}^1 \times \boldsymbol{M}^2_1$ from $\boldsymbol{M}^2$ and $\boldsymbol{M}^1 \times \boldsymbol{M}^3_1$ from $\boldsymbol{M}^3$ yields a new set of equations again possessing the same solution:

$$\boldsymbol{M} \equiv \begin{array}{cccc} 1 & 3 & 2 & 1 \\ 0 & 3 & {}^{-}3 & 6 \\ 0 & {}^{-}5 & 0 & 0 \end{array}$$

The row $\boldsymbol{M}^2$ can now be divided by $\boldsymbol{M}^2_2$ to yield

$$\boldsymbol{M} \equiv \begin{array}{cccc} 1 & 3 & 2 & 1 \\ 0 & 1 & {}^{-}1 & 2 \\ 0 & {}^{-}5 & 0 & 0 \end{array}$$

Then $\boldsymbol{M}^1 \leftarrow \boldsymbol{M}^1 - \boldsymbol{M}^1_2 \times \boldsymbol{M}^2$ and $\boldsymbol{M}^3 \leftarrow \boldsymbol{M}^3 - \boldsymbol{M}^2_3 \times \boldsymbol{M}^3$ yields

$$\boldsymbol{M} \equiv \begin{array}{cccc} 1 & 0 & 5 & {}^{-}5 \\ 0 & 1 & {}^{-}1 & 2 \\ 0 & 0 & {}^{-}5 & 10 \end{array}$$

Again $\boldsymbol{M}^3 \div \boldsymbol{M}^3_3$ yields

$$\boldsymbol{M} \equiv \begin{array}{cccc} 1 & 0 & 5 & {}^{-}5 \\ 0 & 1 & {}^{-}1 & 2 \\ 0 & 0 & 1 & {}^{-}2 \end{array}$$

Finally, $\boldsymbol{M}^1 \leftarrow \boldsymbol{M}^1 - \boldsymbol{M}^1_3 \times \boldsymbol{M}^3$ and $\boldsymbol{M}^2 \leftarrow \boldsymbol{M}^2 - \boldsymbol{M}^2_3 \times \boldsymbol{M}^3$ yields

$$\boldsymbol{M} \equiv \begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & {}^{-}2 \end{array}$$

The first row of this matrix is equivalent to

$$(1 \times \boldsymbol{x}_1) + (0 \times \boldsymbol{x}_2) + (0 \times \boldsymbol{x}_3) - 5 \equiv 0$$

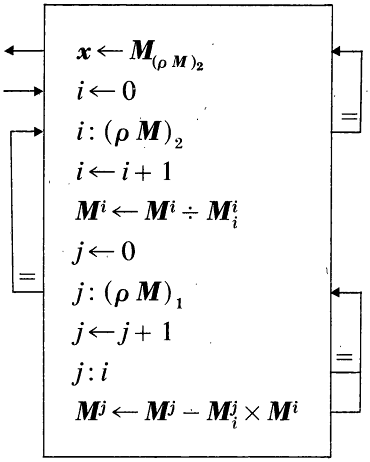
or

$$\boldsymbol{x}_1 \equiv 5$$

Similarly, rows 2 and 3 yield

$$\boldsymbol{x}_2 \equiv 0 \quad \text{and} \quad \boldsymbol{x}_3 \equiv {}^{-}2$$

The foregoing procedure is formalized in Program 4.9 for any matrix $\boldsymbol{M}$. Difficulties can arise in its execution if the element $\boldsymbol{M}^i_i$ in statement 5 is ever zero. Methods of surmounting these difficulties are explored in Exercise 4.27.



```
$x \leftarrow M_{(\rho M)_2}$
$i \leftarrow 0$
$i : (\rho M)_2$
$i \leftarrow i + 1$
$M^i \leftarrow M^i \div M^i_i$
$j \leftarrow 0$
$j : (\rho M)_1$
$j \leftarrow j + 1$
$j : i$
$M^j \leftarrow M^j - M^j_i \times M^i$
```

Program 4.9 Solution of equations $(+/M^i \times x, {}^-1) \equiv 0$

For any matrix M, the *restructuring* function $d \rho M$ produces the vector x such that $x \equiv M^1, M^2, M^3, \ldots$ It is necessary that $d \equiv \times/ \rho M$. For example, if

$$M \equiv \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{matrix}$$

then $12 \rho M \equiv \iota 12$. Moreover, if d is a vector of dimension two such that $\times/ d \equiv \rho x$ then the function $d \rho x$ produces a matrix Q such that $\rho Q \equiv d$ and that $(\times/ d) \rho Q \equiv x$. For example, if $x \equiv 3, 6, 9, 12, 15, 18$, then

$$(2, 3) \rho x \equiv \begin{matrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{matrix} \quad \text{and} \quad (3, 2) \rho x \equiv \begin{matrix} 3 & 6 \\ 9 & 12 \\ 15 & 18 \end{matrix}$$

Finally, if the argument on the right of ρ is a scalar it is extended in the usual way. Hence $n \rho x$ produces a vector of dimension n whose components are each equal to x.

(Do Exercises 4.27–4.41.)

Exercises

4.1 Evaluate $(3, 1, 0, 4) \Pi 2$ by executing Programs 4.1 (a), 4.1 (b), and 4.2.

4.2 Evaluate each of the following:

(a) $(\iota 4)\,\Pi\,3$
(b) $(0, \iota 4)\,\Pi\,3$
(c) $(0 \neq \iota 4)\,\Pi\,3$

4.3 Find the zeros of each of the following polynomials:

(a) $(2, 1, {}^{-}6)\,\Pi\,x$ (c) $(c, 1, 1)\,\Pi\,x$
(b) $(3, 7, 1)\,\Pi\,x$ (d) $\boldsymbol{c}\,\Pi\,x$, where $\rho\,\boldsymbol{c} \equiv 3$

4.4 Determine the vector $\boldsymbol{d}$ such that $\boldsymbol{d}\,\Pi\,x \equiv (\boldsymbol{b}\,\Pi\,x) + \boldsymbol{c}\,\Pi\,x$, where

(a) $\boldsymbol{b} \equiv 3, 1, 0, 2$ and $\boldsymbol{c} \equiv 6, {}^{-}2, 3, 1$
(b) $\boldsymbol{b} \equiv 3, 1, 0, 2$ and $\boldsymbol{c} \equiv {}^{-}3, 2, 2, 2$
(c) $\boldsymbol{b} \equiv 3, 1, 0, 2$ and $\boldsymbol{c} \equiv 2, {}^{-}1, 1, {}^{-}2$
(d) $\boldsymbol{b} \equiv 1, 2, 3, 4$ and $\boldsymbol{c} \equiv 1, 0, 2, 0, 3, 1$
(e) $\boldsymbol{b} \equiv 5, 4, 3, 2, 1$ and $\boldsymbol{c} \equiv 1, 2, 3$

4.5 Determine the vector $\boldsymbol{d}$ such that

$$\boldsymbol{d}\,\Pi\,x \equiv (\boldsymbol{b}\,\Pi\,x) - \boldsymbol{c}\,\Pi\,x$$

for cases (a) and (b) of Exercise 4.4.

4.6 For each of the following cases, determine the vector $\boldsymbol{d}$ such that $\boldsymbol{d}\,\Pi\,x$ is the sum of the given polynomials, after first putting each of the polynomials in the standard form $\boldsymbol{c}\,\Pi\,x$.

(a) $((3 \times x^2) - 6) + x - 2 \times x^5$ and $((2 \times x) - 3) + x^4$
(b) $x^4 + (3 \times x^2) + (2 \times x) - 4$ and $(4 - 3 \times x) + (2 \times x^2) - x^3$

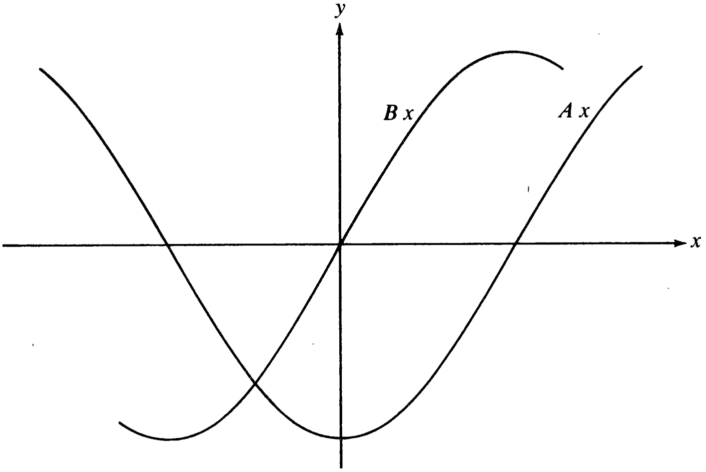
4.7 Use the method of Table 4.3 to determine the product of each of the following pairs of polynomials:

(a) $(3, 1, 4)\,\Pi\,x$ and $(2, 0, 13, 2)\,\Pi\,x$
(b) $(\iota 4)\,\Pi\,x$ and $(⌽\,\iota 4)\,\Pi\,x$
(c) $(1, {}^{-}1)\,\Pi\,x$ and $(\iota 4)\,\Pi\,x$
(d) $(1, 1)\,\Pi\,x$ and $(1, 3, 3, 1)\,\Pi\,x$
(e) $(1, 1)\,\Pi\,x$ and $(1, 4, 6, 4, 1)\,\Pi\,x$
(f) $((3 \times x^2) - 2 \times x) + 6 \times x^4$ and $(2 \times x) + 3 - 2 \times x^3$

4.8 Use the method of Table 4.3 to determine the coefficients of each of the following polynomials:

(a) $((1, 1)\,\Pi\,x) \times (1, 1)\,\Pi\,x$
(b) $((1, 1)\,\Pi\,x) \ast 2$
(c) $((1, 1)\,\Pi\,x) \ast 6$
(d) $((1, 2, 1)\,\Pi\,x) \ast 3$

4.9 Any function $A\,x$ such that $A\,x \equiv A - x$ is called an *even* function; any function $B\,x$ such that $B\,x \equiv -B - x$ is called an *odd* function. The sketch below shows the typical symmetry about the y-axis of an even function $A\ x$ and the typical symmetry about the origin of an odd function $B\ x$.



(a) Prove that any polynomial containing only even powers of the argument x is an even function.
(b) Prove that any polynomial containing only odd powers of the argument x is an odd function.
(c) Choose values for a vector $\boldsymbol{c}$ such that $\boldsymbol{c}\ \Pi\ x$ is neither even nor odd.

4.10 Write a program to determine the coefficients of the polynomial $(\boldsymbol{b}\ \Pi\ x) \times \boldsymbol{c}\ \Pi\ x$.

4.11 What are the relations between the following pairs of polynomials?
(a) $(\iota\ n)\ \Pi\ x$ and $(0\ ,\iota\ n)\ \Pi\ x$
(b) $((1\ ,{}^{-}1)\ \Pi\ x) \times (0 \neq \iota\ n)\ \Pi\ x$ and $(1\ ,-\ n = \iota\ n)\ \Pi\ x$
(c) $((1\ ,\ 1)\ \Pi\ x) \times \boldsymbol{p}\ \Pi\ x$ and $((0\ ,\boldsymbol{p}) + (\boldsymbol{p}\ ,0))\ \Pi\ x$
(d) $(a \times \boldsymbol{c})\ \Pi\ x$ and $a \times \boldsymbol{c}\ \Pi\ x$

4.12 (a) Execute the program of Figure 4.4 for the case

$$\boldsymbol{n} \equiv 6\ ,\ 1\ ,\ 3\ ,\ {}^{-}2\ ,\ 7\ ,\ 4 \text{ and } \boldsymbol{d} \equiv 2\ ,\ {}^{-}2\ ,\ 1.$$

(b) Check the result of part (a) by multiplication.
(c) Perform the synthetic division of part (a) using the tabular arrangement of Figure 4.4 (c).

4.13 Use the arrangement of Figure 4.4 (c) to perform the following synthetic divisions:

(a) (7 , 6 , 1 , ¯3 , ¯4) Π x by (1 , 2 , 1) Π x
(b) (7 , 6 , 1 , ¯3 , ¯4) Π x by (¯3 , 1) Π x
(c) (1 , 7 , 10 , ¯3 , 3 , 2) Π x by (1 , 4 , ¯3 , 2) Π x
(d) $((6\times x^5) - 2\times x^3) + (10\times x^4) - 2$ by $(x^3 - 3\times x) + 6$

4.14 Use the results of Exercise 4.13 (b) to determine the value of (7 , 6 , 1 , ¯3 , ¯4) Π 3, and check your result by using Program 4.2.

4.15 Show that $9 \mid +/\,\boldsymbol{c} \equiv 9 \mid \boldsymbol{c}\ \Pi\ 10$ (see Exercise 3.32).

4.16 (a) Write a program to determine $\boldsymbol{p}$ such that $\boldsymbol{p}\ \Pi\ x \equiv (1 + x) * n$.
(b) Execute the program of part (a) for the case $n \equiv 5$ and compare the result with Table 4.5.
(c) Use the program of part (a) to determine $\beta\,6$.

4.17 Use Table 4.5 in the following calculations:
(a) Determine to the nearest cent the present value of an investment of \$100 made five years ago and left to accumulate at the rate of 4 percent per annum compounded annually.
(b) Determine to the nearest cent the present value of an investment of \$100 made five years ago and left to accumulate at the rate of 4 percent per annum compounded semiannually.
(c) Determine the ratio between the frequencies of two piano notes separated by twelve half tones, where the ratio between successive half tones is approximately 1.06. (Extend Table 4.5 as required and compute to three decimal places.)
(d) Use the results of part (c) to determine whether the precise ratio between frequencies of successive half tones is greater than or less than 1.06. (Note that twelve half tones constitute one octave.)

4.18 Evaluate the following expressions, using any means you wish to simplify the work:
(a) ((5 , 4 , 3 , 2 , 1) Π 3) − (2 , 4 , 3 , 2) Π 3
(b) (1 , 5 , 10 , 10 , 5 , 1) Π 2
(c) $+/\,\iota\,10$
(d) ((5 , 4 , 3 , 2 , 1) Π 2) + (⌽ 5 , 4 , 3 , 2 , 1) Π 2

4.19 Use Table 4.5 to determine the coefficients of each of the following polynomials:
(a) $(x + 2) * 2$
(b) $(x + 2) * 3$
(c) $(x + 2) * 5$

4.20 Use one synthetic division to check all three results of Exercise 4.19.

4.21 (a) Show that

$$((1\,,{}^{-}1)\,\Pi\,x)\ast 5\equiv(1\,,{}^{-}5\,,10\,,{}^{-}10\,,5\,,{}^{-}1)\,\Pi\,x$$

(b) Show that $((1\,,{}^{-}1)\,\Pi\,x)\ast n\equiv(\boldsymbol{v}\times\beta\,n)\,\Pi\,x$, where

$$\boldsymbol{v}\equiv{}^{-}1\ast 0\,,\iota\,n\equiv 1\,,{}^{-}1\,,1\,,{}^{-}1\,,\ldots$$

(c) Show that $\boldsymbol{c}\,\Pi\,1\equiv +/\,\boldsymbol{c}$.

(d) Use the results of parts (c) and (b) to show that

$$+/\,\boldsymbol{v}\times\beta\,n\equiv 0$$

(e) Verify the result of part (d) for each row of Table 4.5.

4.22 (a) Compute and compare the values of the functions $(0\,,.5\,,.5)\,\Pi\,x$ and $+/\,\iota\,x$ for $x\equiv 1, 2, 3, 4, 5, 6$, and 7.

(b) Prove that the functions of part (a) are equal for all integer values of x. (Consider $+/\,⌽\,\iota\,x$ and $(\iota\,x)+⌽\,\iota\,x$.)

(c) Assuming that $\rho\,\boldsymbol{c}\equiv 3$, determine a polynomial $\boldsymbol{c}\,\Pi\,x$ that fits the function $+/\,\iota\,x$ at three points.

4.23 (a) Tabulate the function $+/\,(\iota\,x)\ast 2$ for $x\equiv 1\,,2\,,3\,,4\,,5\,,6\,,7\,,8$ and determine a polynomial of degree three that fits the function at the first four points.

(b) Verify that the derived polynomial fits the function at a number of further points.

4.24 (a) Determine a polynomial to fit the function $+/\,(\iota\,x)\ast 3$.

(b) Square the polynomial of Exercise 4.22 (a) and compare with the polynomial derived in part (a) above.

4.25 (a) Make tables of the values of each of the following functions for values of n from zero to five:

(i) $+/\,\iota\,n$

(ii) $(0\,,.5\,,.5)\,\Pi\,n$

(iii) $-/\,(\iota\,n)\ast 2$

(b) From the results of part (a) makes a quick calculation of what you would expect the value of $-/\,(\iota\,20)\ast 2$ to be.

4.26 The result proved in Exercise 4.22 can also be proved by the use of *mathematical induction* as follows. Assume that the relation $(0\,,.5\,,.5)\,\Pi\,x\equiv +/\,\iota\,x$ holds for x equal to some integer k. Then $(0\,,.5\,,.5)\,\Pi\,k\equiv +/\,\iota\,k$.

But

$$\begin{aligned}(0\,,.5\,,.5)\ \Pi\ k+1 &\equiv \frac{(k+1)+(k+1)^2}{2}\\ &\equiv \frac{k+k^2}{2}+k+1\\ &\equiv ((0\,,.5\,,.5)\ \Pi\ k)+(k+1)\\ &\equiv (+/\,\iota\, k)+k+1\\ &\equiv +/\,\iota\, k+1\end{aligned}$$

Hence the assumption that $(0\,,.5\,,.5)\ \Pi\ x \equiv +/\,\iota\, x$ holds for $x \equiv k$ leads to the conclusion that it must also hold for $x \equiv k+1$. But the relation obviously holds for $x \equiv 1$, since

$$(0\,,.5\,,.5)\ \Pi\ 1 \equiv 1 \text{ and } +/\,\iota\, 1 \equiv 1$$

Therefore $(0\,,.5\,,.5)\ \Pi\ x \equiv +/\,\iota\, x$ for $x \equiv 1$. Since this is true for $x \equiv 1$, it is also true for $x \equiv 1+1$ or 2. Since true for 2, it is also true for 3, and so on for all succeeding integers.

The method of mathematical induction can be stated as follows: (1) if the assumption that the values of two functions of x are equal for $x \equiv k$ (k an integer) implies that they are also equal for $x \equiv k+1$, and (2) if the functions can be shown to agree for some integral value of x, then they must agree for all succeeding integral values of x.

(a) Use mathematical induction to show that the polynomial derived in Exercise 4.23 (a) agrees with the function $+/\ (\iota\, x) * 2$ for all integral values of x.

(b) Show that the polynomial of Exercise 4.24 agrees with the function $+/\ (\iota\, x) * 3$ for all integral values of x.

4.27 Program 4.9 fails if the *pivot row* $\boldsymbol{M}^i$ has a zero component in its ith position $\boldsymbol{M}^i_i$ (called the *pivot element*), since the division by $\boldsymbol{M}^i_i$ in statement 5 cannot then be carried out. However, since the equations can be reordered (that is, $\boldsymbol{M}^k$ and $\boldsymbol{M}^i$ can be interchanged) without affecting the solution, the offending pivot row $\boldsymbol{M}^i$ can be exchanged with some other row $\boldsymbol{M}^k$ such that $\boldsymbol{M}^k_i \neq 0$. However, the exchange can be made only with some row $\boldsymbol{M}^k$ that has not already served as a pivot row; that is, k must exceed i.

(a) Rewrite Program 4.9 so as to avoid the difficulty of a zero pivot element.

(b) Rewrite Program 4.9 so as to choose the pivot row in the ith stage as that row with the largest (in absolute value) pivot element among the eligible rows. This is one of the best procedures for minimizing the accumulation of round-off error in the solution.

4.28 Let $\boldsymbol{g}$ be a given set of argument values for a function F, and let $\boldsymbol{r}$ be the corresponding set of function values. In other words, $\boldsymbol{r} = F\,\boldsymbol{g}$. Write a program that will determine the coefficients $\boldsymbol{c}$ of a polynomial that fits the function F at all the points $(\boldsymbol{g}_i\,, F\,\boldsymbol{g}_i)$.

4.29 (a) Write a program that will determine a polynomial of degree n that fits the function F at equally spaced points in the interval from $\boldsymbol{a}_1$ to $\boldsymbol{a}_2$, including the endpoints.

(b) Execute the program of part (a) for

$$F\,x \equiv 1 \div (1\,,1\,,1)\,\Pi\,x$$

for $\boldsymbol{a} \equiv 0\,,1$ and $n \equiv 3$.

4.30 If $\boldsymbol{c}\,\Pi\,z \equiv 0$, then z is said to be a *zero* of the polynomial with coefficients $\boldsymbol{c}$. If z is a zero of $\boldsymbol{c}\,\Pi\,x$, then $(x - z)$ is a factor of $\boldsymbol{c}\,\Pi\,x$, and hence there is a polynomial $\boldsymbol{d}$ such that $\boldsymbol{c}\,\Pi\,x \equiv (x - z) \times \boldsymbol{d}\,\Pi\,x$. If $\boldsymbol{c}\,\Pi\,x$ contains the factor $(x - z) \ast k$ but not the factor $(x - z) \ast k + 1$, then z is said to be a *zero of multiplicity k*.

Write a program to determine the multiplicity of a zero z of the polynomial $\boldsymbol{c}\,\Pi\,x$.

4.31 Let $\boldsymbol{z}$ be the vector of all the zeros of the polynomial $\boldsymbol{c}\,\Pi\,x$ (any zero of multiplicity k appears k times in $\boldsymbol{z}$).

(a) What is the degree of $\boldsymbol{c}\,\Pi\,x$?

(b) Write a program to determine $\boldsymbol{c}$ as a function of $\boldsymbol{z}$.

4.32 Let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ be numbers such that $(\boldsymbol{c}\,\Pi\,\boldsymbol{a}_1) < 0$ and $(\boldsymbol{c}\,\Pi\,\boldsymbol{a}_2) > 0$. Write a program to determine z as one zero of the polynomial $\boldsymbol{c}\,\Pi\,x$ to within a specified tolerance $\boldsymbol{a}_3$; that is, determine z so that the absolute value of $\boldsymbol{c}\,\Pi\,z$ does not exceed $\boldsymbol{a}_3$.

4.33 (a) Prove that $(0 \neq \iota\,n)\,\Pi\,x \equiv (1 - x \ast n) \div (1 - x)$.

(b) Write a program to determine z as the value of $(0 \neq \iota\,n)$ $\Pi\,x$ using the result of part (a).

(c) Determine the sum of the first six terms of the sequence $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

4.34 A vector $\boldsymbol{s}$ of dimension $\rho\,\boldsymbol{c}$ *separates* the zeros of $\boldsymbol{c}\,\Pi\,x$ if, for each value of i, there is exactly one zero of $\boldsymbol{c}\,\Pi\,x$ between $\boldsymbol{s}_i$ and $\boldsymbol{s}_{i+1}$. Write a program that determines $\boldsymbol{z}$ as the vector of zeros of $\boldsymbol{c} \,\Pi\, x$ to within a tolerance .0001 (that is $.0001 \geq |\boldsymbol{c} \,\Pi\, \boldsymbol{z}_i|$).

4.35 Let $\boldsymbol{p} \,\Pi\, x \equiv ((\iota\, n) \,\Pi\, x) \times (\Phi\, \iota\, n) \,\Pi\, x$.

(a) Compute the value of $\boldsymbol{p}$ for each of the cases $n = 2, 3, 4, 5$.

(b) Prove that $+/\boldsymbol{p} \equiv (+/\iota\, n) * 2$.

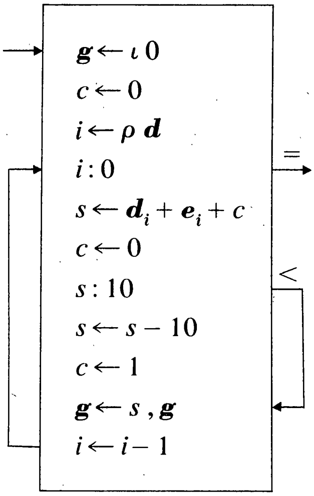
(c) Prove that $\lceil/\boldsymbol{p} \equiv +/(\iota\, n) * 2$.

4.36 If $m \equiv 1776$ and if $\boldsymbol{d}$ is a vector whose components are the successive decimal digits of m (that is, $\boldsymbol{d} \equiv 1, 7, 7, 6$), then $\boldsymbol{d}$ will be called a base-10 vector representation of m, since $m \equiv (\Phi\, \boldsymbol{d}) \,\Pi\, 10$. If $n \equiv 1860$, if $\boldsymbol{e}$ is the representation of n, and if $\boldsymbol{f} \equiv \boldsymbol{d} + \boldsymbol{e}$, then $m + n \equiv (\Phi\, \boldsymbol{f}) \,\Pi\, 10$, since

$$(\Phi\, \boldsymbol{f}) \,\Pi\, 10 \equiv (6, 13, 15, 2) \,\Pi\, 10 \equiv 6 + 130 + 1500 + 2000 \equiv 3636$$

However, $\boldsymbol{f} \equiv (2, 15, 13, 6)$ is *not* the normal representation of the sum of m and n because it possesses components that exceed *nine*. The normal representation $\boldsymbol{g} \equiv (3, 6, 3, 6)$ can be obtained from $\boldsymbol{f}$ by "carrying," that is, by subtracting *ten* from any component that exceeds *nine* and compensating by adding *one* to the component to the left of it.

(a) Execute the program below for the values of $\boldsymbol{d}$ and $\boldsymbol{e}$ used above and verify that it yields the correct representation of the sum of the numbers represented by $\boldsymbol{d}$ and $\boldsymbol{e}$.



```
$g \leftarrow \iota 0$
$c \leftarrow 0$
$i \leftarrow \rho d$
$i : 0$
$s \leftarrow d_i + e_i + c$
$c \leftarrow 0$
$s : 10$
$s \leftarrow s - 10$
$c \leftarrow 1$
$g \leftarrow s, g$
$i \leftarrow i - 1$
```

(b) Show that the program does not give the correct result for the case $\boldsymbol{e} \equiv 7, 6, 1$ and $\boldsymbol{d} \equiv 4, 5, 7$, and modify the program to correct this defect.

(c) Further modify the program to permit argument $\boldsymbol{d}$ to have a dimension different from argument $\boldsymbol{e}$.

4.37 (i) Rewrite the programs of Exercise 4.36, using relational functions and the residue to simplify the programs as much as possible.

(ii) Modify the programs of part (i) to perform addition in any specified base b rather than in base 10.

4.38 Write a program to determine $\boldsymbol{p}$ as the representation of the product of the numbers represented by $\boldsymbol{d}$ and $\boldsymbol{e}$, that is, $(⌽\, \boldsymbol{p})\, \Pi\, 10 \equiv ((⌽\, \boldsymbol{d})\, \Pi\, 10) \times (⌽\, \boldsymbol{e})\, \Pi\, 10$.

4.39 (a) Prove that if $\boldsymbol{d}$ is the decimal representation of the integer m (that is, $m \equiv (⌽\, \boldsymbol{d})\, \Pi\, 10$), then $9 | m \equiv 9 | +/\, \boldsymbol{d}$ (see Exercise 3.32).

(b) Show that m is divisible by 9 if and only if $+/\, \boldsymbol{d}$ is divisible by 9.

(c) Show that m is divisible by 11 if and only if $-/\, \boldsymbol{d}$ is divisible by 11.

(d) Extend the results of parts (a), (b), and (c) to divisibility by $(b - 1)$ and $(b + 1)$ in the representation in any specified base b.

4.40 (a) Write a program to determine $\boldsymbol{d}$ as the base-10 vector representation of the argument m.

(b) Write a program to determine $\boldsymbol{g}$ as the base-b representation of the number whose base-10 representation is $\boldsymbol{d}$.

4.41 Since $(x + 1) \ast 5 \equiv (1, 5, 10, 10, 5, 1)\, \Pi\, x$, then $((x + 1) \ast 5) - x \ast 5 \equiv (1, 5, 10, 10, 5)\, \Pi\, x$. Applying this identity for $x = 1, 2, 3, \ldots,$ and n yields

$$2^5 - 1^5 \equiv 1 + (5 \times 1) + (10 \times 1^2) + (10 \times 1^3) + (5 \times 1^4)$$
$$3^5 - 2^5 \equiv 1 + (5 \times 2) + (10 \times 2^2) + (10 \times 2^3) + (5 \times 2^4)$$
$$4^5 - 3^5 \equiv 1 + (5 \times 3) + (10 \times 3^2) + (10 \times 3^3) + (5 \times 3^4)$$
$$\cdot \quad \cdot$$
$$(n + 1)^5 - n^5 \equiv 1 + (5 \times n) + (10 \times n^2) + (10 \times n^3) + (5 \times n^4)$$

Adding these identities (and simplifying as much as possible on the left) yields

$$((n + 1) \ast 5) - 1 \equiv n + (5 \times +/\, \iota\, n) + (10 \times +/\, (\iota\, n) \ast 2) + (10 \times +/\, (\iota\, n) \ast 3) + 5 \times +/\, (\iota\, n) \ast 4$$

Solving for $+/\, (\iota\, n) \ast 4$ yields

$$5 \times +/\, (\iota\, n) \ast 4 \equiv ((n + 1) \ast 5) - (1 + n + (5 \times +/\, \iota\, n)) \\ + (10 \times +/\, (\iota\, n) \ast 2) + 10 \times +/\, (\iota\, n) \ast 3$$

Substituting known polynomials (see Exercise 4.24) for the sums on the right and adding yields

$$\begin{array}{rrrrrr}
1 & 5 & 10 & 10 & 5 & 1 \\
-1 & & & & & \\
0 & -1 & & & & \\
0 & -2.5 & -2.5 & & & \\
0 & -\frac{10}{6} & -5 & -\frac{10}{3} & & \\
0 & 0 & -2.5 & -5 & -2.5 & \\
\hline
0 & -\frac{1}{6} & 0 & \frac{10}{6} & 2.5 & 1
\end{array}$$

Finally, therefore,

$$+/\, (\iota\, n) \ast 4 \equiv ((0\, , {}^{-}1\, , 0\, , 10\, , 15\, , 6)\ \Pi\ n) \div 30$$

(a) For the cases $n = 0, 1, 2, 3,$ and 4, verify that the foregoing polynomial is equal to the sum of the first n integers each raised to the fourth power.

(b) Use the same method to find a polynomial for the sum of the fifth powers of the first n integers.

(c) Write a program to determine $\boldsymbol{p}$ as the vector of coefficients of the polynomial that fits the function $+/\, (\iota\, n) \ast m$.

Chapter Five

The Slope Function

The *slope* of a straight line is defined as the ratio of the vertical rise to the horizontal distance between any pair of points on the line. For example, the slopes of the straight lines of Figure 5.1 are 2, 1, .5, $^{-}2$, and 0, as shown. The graph of the function $F\,x \equiv b + a \times x$ is a straight line with slope a. Moreover, if $r \neq p$, then the straight line through the points $(p\,,q)$ and $(r\,,s)$ has slope $\frac{s-q}{r-p}$.

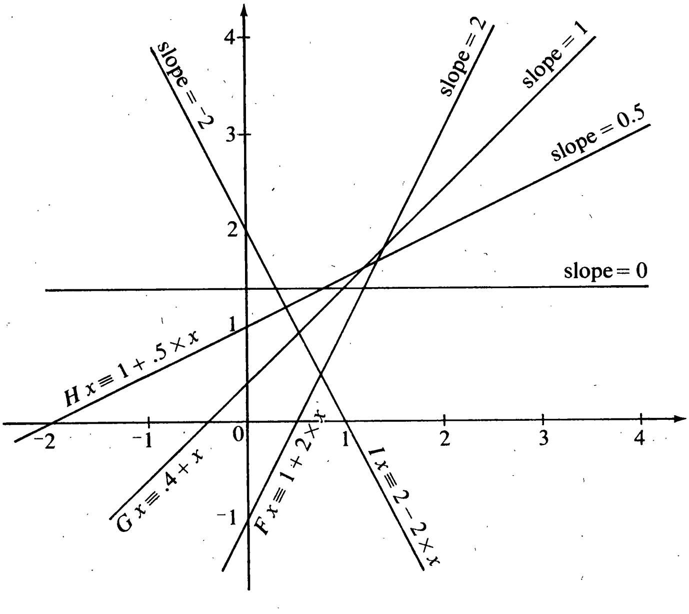
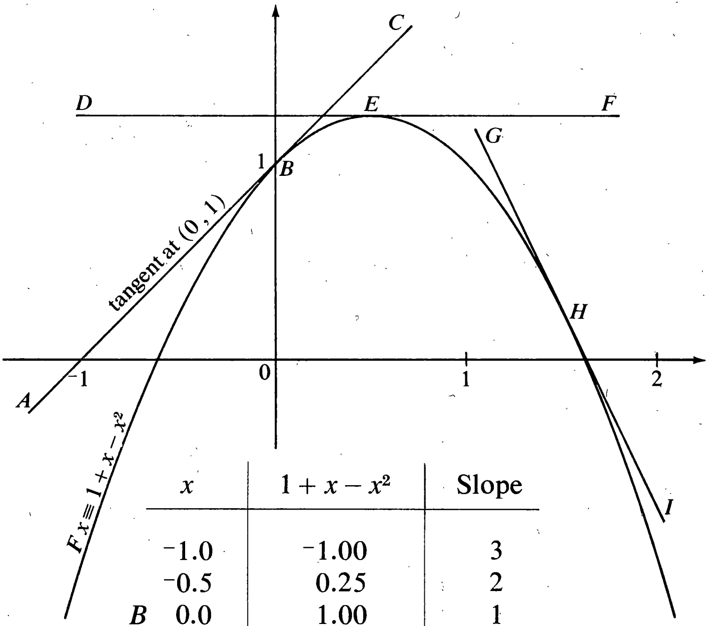


Figure 5.1 Straight lines with various slopes

The graph of a function is not necessarily a straight line and therefore may not have a fixed slope. However, at each point on a curve there is a tangent† line that does have a fixed slope. This is illustrated in Figure 5.2 for the graph of the function $F\,x \equiv 1 + x - x^2$. Thus, at the point $(0\,, F\,0)$ the tangent is the line *ABC*, whose slope is 1. Similarly, the tangent at the point $(.5\,, 1.25)$ is the line *DEF* with slope 0, and the tangent at $(1.5\,, .25)$ is the line *GHI* with slope $^{-}2$.



	x	$1 + x - x^2$	Slope
	$^{-}1.0$	$^{-}1.00$	3
	$^{-}0.5$	0.25	2
B	0.0	1.00	1
E	0.5	1.25	0
	1.0	1.00	$^{-}1$
H	1.5	0.25	$^{-}2$
	2.0	$^{-}1.00$	$^{-}3$

Figure 5.2 Slopes of tangents to a curve

Knowledge of the slope of the graph of a function can be very useful. In Figure 5.2, for example, the fact that the tangent *DEF* is

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†The terms *tangent, chord,* and *secant* are used for any curve in the same way that they are used for the circle. A *secant* is a straight line that intersects the curve in two (or more) points, a *chord* is that segment of the secant between the points of intersection, and a *tangent* is the limiting position of the secant as one point of intersection approaches the other (that is, the tangent touches the curve at one point).

horizontal (that is, has slope 0) makes it clear that the point of tangency (.5 , 1.25) is a maximum point of the function $F\,x \equiv 1 + x - x^2$. Similarly, the fact that the slope at B is positive indicates that the function is increasing in the vicinity of that point, and the fact that the slope at H is $^-2$ indicates that the function is decreasing rapidly in the vicinity of that point.

The slope of a polynomial (and indeed of any of the elementary functions) can be determined at *every point* on its graph by simple methods to be developed in this chapter; in other words, the slope at $(x\,, F\,x)$ is itself a *function* of x. For example, the slope of the function $F\,x \equiv 1 + x - x^2$ of Figure 5.2 is equal to $1 - 2 \times x$ for any value of x, and the slope of the function $G\,x \equiv x + x^2 - x^3$ of Figure 5.3 is the function $H\,x \equiv 1 + (2 \times x) - 3 \times x^2$, also shown in the figure. The function $H\,x$ is called the *slope function* of the function $G\,x$, or simply the *slope* of $G\,x$.

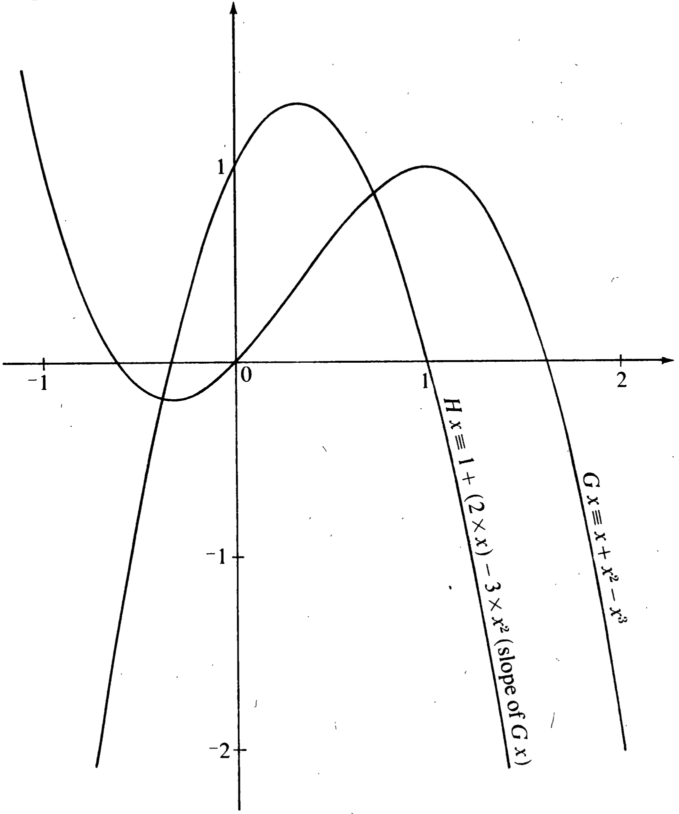


Figure 5.3 A function $G\,x$ and its slope function $H\,x$

The slope of a function is important in the study of the function; the elementary functions can, in fact, be defined in terms of their slopes. For example, the function $E\,x$ graphed in Figure 5.4 is defined by simply requiring that its value be 1 when $x = 0$ and that its slope at each point be equal to the value of the function itself at the point.

x	$E\,x$
-3	0.050
-2	0.135
-1	0.368
0	1.000
1	2.718
2	7.389
3	20.086

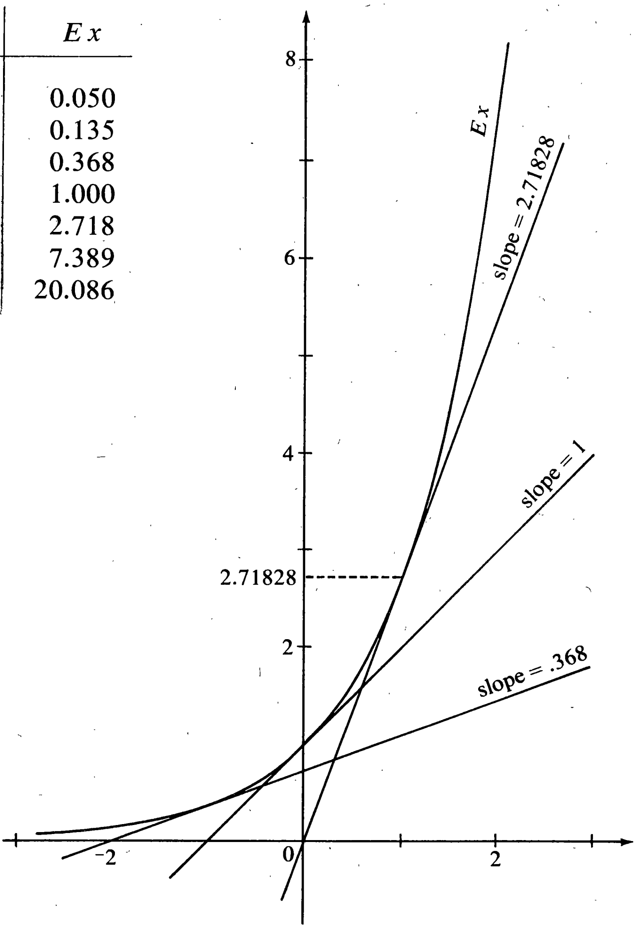


Figure 5.4 The function $E\,x$ whose slope function is equal to $E\,x$

It is perhaps the most useful function in applied mathematics. Moreover, a study of the slope of an elementary function leads to simple means for evaluating the function.

(Do Exercises 5.1 – 5.2.)

The Secant Slope of a Function

In order to develop means for determining the slope of a function, it is necessary to begin with a careful definition of the slope of a tangent. The slope of the secant through the points $(x, F\,x)$ and $((x+s), F\,x+s)$ will be denoted by $(D_s F)\,x$. Thus

$$(D_s F)\,x \equiv \frac{(F\,x+s) - F\,x}{(x+s) - x} \equiv \frac{(F\,x+s) - F\,x}{s} \qquad (5.1)$$

as illustrated in Figure 5.5 for the function $F\,x \equiv x^2 - (x^3 \div 3)$ and for $s = 1$.

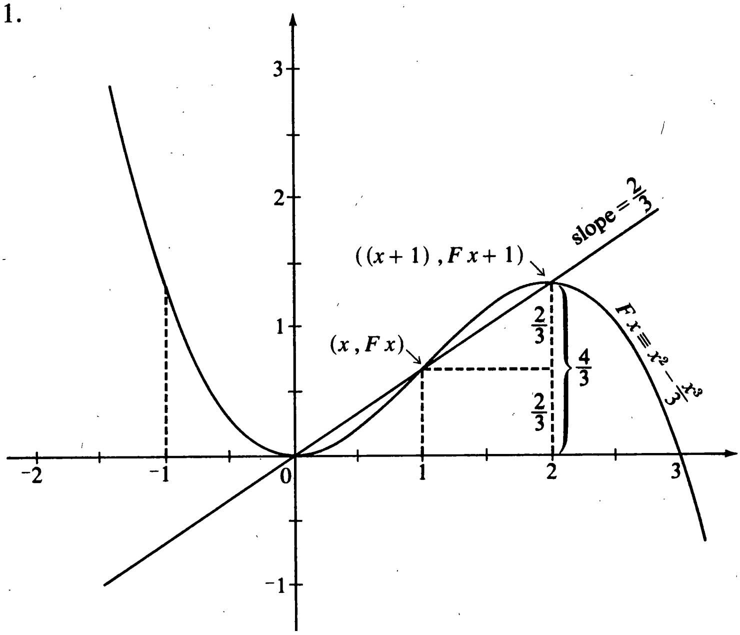


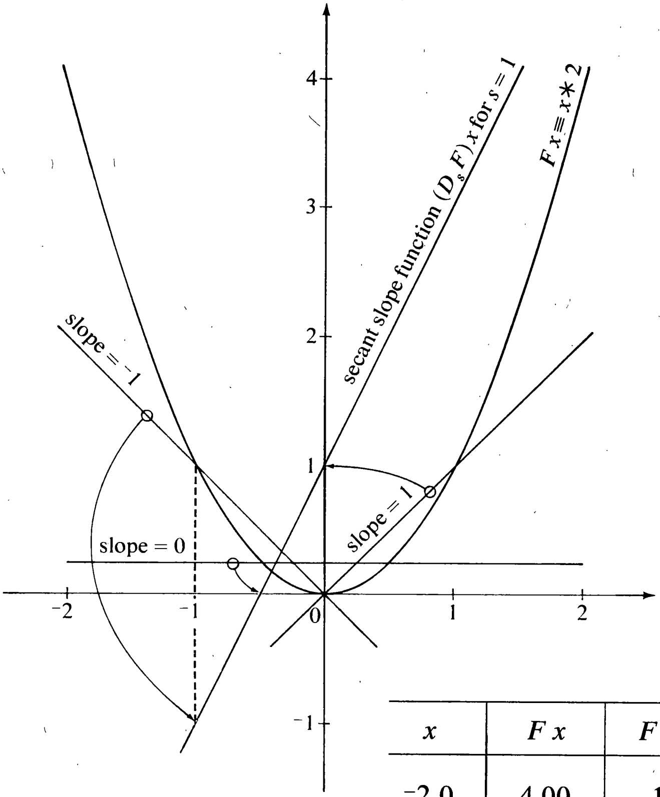
Figure 5.5 The secant slope $(D_s F)\,x$ for $s = 1$ and $x = 1$

For any chosen function $F\,x$ and for any fixed value of s, it is clear that the value of $(D_s F)\,x$ is determined for every value of x. In other words, $(D_s F)\,x$ is itself a function of x; it will be called the *secant slope function* of F, or the *secant slope* of F. Figure 5.6 shows an example in which $F\,x \equiv x * 2$ and the corresponding secant slope function $(D_1 F)\,x \equiv 1 + 2 \times x$, for the case $s = 1$. This expression for the secant slope of $F\,x$ is easily derived by setting $s = 1$ in Equation 5.1 to obtain

$$(D_1 F)\,x \equiv \frac{(F\,x+1) - F\,x}{1}$$

and then using the fact that $F\ x$ is the function $x*2$, thus:

$$\begin{aligned}(D_1\,F)\ x &\equiv ((x+1)*2) - x*2\\ &\equiv 1 + (2\times x) + (x*2) - x*2\\ &\equiv 1 + 2\times x\end{aligned}$$



x	$F\ x$	$F\ x + s$	$(D_s\,F)\ x$
¯2.0	4.00	1.00	¯3
¯1.5	2.25	0.25	¯2
¯1.0	1.00	0.00	¯1
¯0.5	0.25	0.25	0
0.0	0.00	1.00	1
0.5	0.25	2.25	2
1.0	1.00	4.00	3
1.5	2.25	6.25	4
2.0	4.00	9.00	5

Figure 5.6 The secant slope function $(D_s\,F)\ x$ for $s = 1$

More generally, for any nonzero value of s:

$$(D_s F)\ x \equiv \frac{((x+s) \ast 2) - x \ast 2}{s}$$
$$\equiv \frac{(s \ast 2) + 2 \times x \times s}{s}$$

Hence, for $F\ x \equiv x \ast 2$,

$$(D_s F)\ x \equiv s + 2 \times x \tag{5.2}$$

Table 5.7 shows values of the secant slope of F for various points and for various values of s. It is clear from the table (as well as from Equation 5.2) that the secant slope approaches a limiting value as s approaches 0. Since s is the spacing (along the x-axis) of the two points that determine the secant, the geometrical meaning is that the slope of the secant approaches the slope of the tangent as its limiting value as s approaches 0. Hence, for $s = 0$, Equation 5.2 gives the slope of the *tangent* to the curve $F\ x \equiv x \ast 2$ at any point x.

x	$(D_s F)\ x$					
	$s = 1$	$s = 0.1$	$s = 0.01$	$s = 0.001$	$s = 0.0001$	$s = 0$
¯2.0	¯3	¯3.9	¯3.99	¯3.999	¯3.9999	¯4
¯1.5	¯2	¯2.9	¯2.99	¯2.999	¯2.9999	¯3
¯1.0	¯1	¯1.9	¯1.99	¯1.999	¯1.9999	¯2
¯0.5	0	¯0.9	¯0.99	¯0.999	¯0.9999	¯1
0.0	1	0.1	0.01	0.001	0.0001	0
0.5	2	1.1	1.01	1.001	1.0001	1
1.0	3	2.1	2.01	2.001	2.0001	2
1.5	4	3.1	3.01	3.001	3.0001	3
2.0	5	4.1	4.01	4.001	4.0001	4

Table 5.7 Secant slopes for $F\ x \equiv x \ast 2$

The slope of the tangent to the graph of any function F at the point $(x, F\ x)$ will therefore be defined as the value approached by the secant slope $(D_s F)\ x$ as the spacing s approaches 0. This tangent slope function will be denoted by $(D\ F)\ x$ and will be called the *slope function* of F. Because the slope function $(D\ F)\ x$ is *derived* from F, it is often called the *derivative* of F; hence the choice of the symbol D to denote it.

Although the denominator of the right-hand side of Equation 5.1 approaches 0 as s approaches 0, the numerator also approaches 0 in such a manner that the ratio approaches a fixed limiting value. For any particular function F this value is obtained by cancelling the factor s from both numerator and denominator before s is equated to 0. The slope functions of all the elementary functions can be obtained in this way.

(Do Exercises 5.3–5.5.)

The Slope of the Exponential Function $x \ast n$

Equation 5.1 was used earlier to derive the slope function of the function $x \ast 2$. It will now be used to determine the slope function of the function $F\,x \equiv x \ast n$ for other nonnegative integer values of n.

For $n = 3$, Equation 5.1 yields

$$\begin{aligned}(D_s F)\,x &\equiv \frac{((x+s) \ast 3) - x \ast 3}{s}\\ &\equiv (3 \times x^2) + (3 \times s \times x) + s^2\end{aligned}$$

Thus, for $s = 0$,

$$(D\,F)\,x \equiv 3 \times x^2$$

For a general value of n, the secant slope of $F\,x \equiv x \ast n$ becomes

$$(D_s F)\,x \equiv \frac{((x+s) \ast n) - x \ast n}{s} \tag{5.3}$$

The binomial theorem (Equation 4.1 (b)) can be applied to obtain an expression for the first term of Equation 5.3:

$$(x+s) \ast n \equiv ((\beta\,n) \times \Phi\, x \ast 0\,,\iota\,n)\,\Pi\,s$$

Since the first and second components of $\beta\,n$ are obviously 1 and n respectively (see Table 4.5), then $\beta\,n \equiv (1\,,n\,,\boldsymbol{c})$, where $\boldsymbol{c}$ is the vector of the remaining $n-1$ components. Hence

$$\begin{aligned}(x+s) \ast n &\equiv (1\,,n\,,\boldsymbol{c}) \times \Phi\, x \ast 0\,,\iota\,n)\,\Pi\,s\\ &\equiv ((x \ast n)\,,(n \times x \ast n-1)\,,\boldsymbol{c} \times \Phi\, x \ast 0\,,\iota\,n-2)\,\Pi\,s\\ &\equiv (x \ast n) + s \times ((n \times x \ast n-1)\,,\boldsymbol{c} \times \Phi\, x \ast 0\,,\iota\,n-2)\,\Pi\,s\end{aligned}$$

Substituting this result in Equation 5.3, cancelling the terms involving $x \ast n$, and performing the division by s yields

$$(D_s F)\,x \equiv ((n \times x \ast n-1)\,,\boldsymbol{c} \times \Phi\, x \ast 0\,,\iota\,n-2)\,\Pi\,s$$

The slope function $(D\ F)\ x$ is now obtained by evaluating the expression for the secant slope at $s=0$. This value is clearly the constant term of the preceding polynomial in s. Therefore, for $F\ x \equiv x \ast n$,

$$(D\ F)\ x \equiv n \times x \ast n-1 \tag{5.4}$$

This general result can be compared with previous results by applying it for particular values of n:

$$\begin{array}{ll} F\ x \equiv x \ast 3 & (D\ F)\ x \equiv 3 \times x \ast 2 \\ F\ x \equiv x \ast 2 & (D\ F)\ x \equiv 2 \times x \ast 1 \equiv 2 \times x \\ F\ x \equiv x \ast 1 \equiv x & (D\ F)\ x \equiv 1 \times x \ast 0 \equiv 1 \\ F\ x \equiv x \ast 0 \equiv 1 & (D\ F)\ x \equiv 0 \times x \ast {}^{-}1 \equiv 0 \end{array}$$

The first two results agree with those previously obtained for $x \ast 3$ and $x \ast 2$. The third case $(F\ x \equiv x \ast 1 \equiv x)$ is clearly the equation of the straight line with slope 1. The fourth case is the function having the constant value 1, and its slope is clearly 0.

The method just applied to the function $x \ast n$ could be applied to determine the slopes of more complex functions such as $(x^3 - x^2) + 5$ or $\boldsymbol{d} \Pi\, x$. It will prove simpler and more efficient, however, to determine general expressions for the slopes of functions such as $(F\ x) + G\ x$ and $(F\ x) \times G\ x$ in terms of the slopes of $F\ x$ and $G\ x$. Then these results can be applied in a simple manner to obtain the slopes of a wide range of interesting functions. In order to do this it will be convenient to introduce an abbreviated notation for composite functions.

(Do Exercise 5.6.)

Notation for Composite Functions

A monadic function such as

$$H\ x \equiv (2 \lceil x) \times 3 \lceil x$$

which is composed of (that is, defined in terms of) other functions is called a *composite* function. It will be defined by the abbreviated expression

$$H \equiv (2 \lceil) \times 3 \lceil$$

obtained by deleting all occurrences of the argument x. The original expression can be reestablished by inserting the symbols for the argument x. Since a legitimate expression can be obtained only by inserting arguments in precisely the places from which they were dropped, the abbreviated expression is *unambiguous* and therefore provides a definition of the function H.

Similarly, if A and B are two monadic functions, the abbreviated expression

$$H \equiv (A) \times B$$

can be expanded in only one way, namely,

$$H\,x \equiv (A\,x) \times B\,x$$

The parentheses around A are essential, since the expression $P \equiv A \times B$ is an abbreviation for the quite different expression

$$P\,x \equiv A\,x \times B\,x$$

which, because of the right-to-left convention, is equivalent to

$$P\,x \equiv A\,(x \times B\,x)$$

The abbreviated expression $Q \equiv A\,B$ must be interpreted as $Q\,x \equiv A\,B\,x$. For example, if $A\,x \equiv 2 * x$, and $B\,x \equiv 3 \times x - 2$, then $Q\,x \equiv 2 * 3 \times x - 2$.

The use of any symbol (such as $-$ or $|$) that denotes either a monadic or a dyadic function can lead to ambiguity, since $H \equiv A - B$ could be interpreted as either

$$H\,x \equiv A\,(-B\,x) \quad \text{(monadic)}$$

or

$$H\,x \equiv A\,(x - B\,x) \quad \text{(dyadic)}$$

To avoid ambiguity, the monadic interpretation will be adopted whenever an expression permits it. The dyadic case can always be indicated by inserting empty parentheses. For example, $H \equiv A\,(\quad) - B$ would necessarily be interpreted as $H\,x \equiv A\,(x) - B\,x$, which is equivalent to $A\,x - B\,x$.

An abbreviated expression can sometimes be made more readable by adding redundant parentheses. For example, $F \equiv \times$ can also be written as $F \equiv (\quad) \times (\quad)$, since both are equivalent to $F\,x \equiv x \times x$. Similarly, $F \equiv \qquad$ and $F \equiv (\quad)$ are both equivalent to $F\,x \equiv x$.

(Do Exercises 5.7–5.8.)

The Slope of the Sum of Two Functions

The function $(F) + G$ is called the *sum* of F and G. The slope of the function

$$H\,x \equiv (F\,x) + G\,x$$

is easily determined by applying Equation 5.1:

$$\begin{aligned}(D_s H)\,x &\equiv \frac{((F\,x+s)+G\,x+s)-((F\,x)+G\,x)}{s}\\ &\equiv \frac{((F\,x+s)-F\,x)}{s}+\frac{(G\,x+s)-G\,x}{s}\\ &\equiv ((D_s F)\,x)+(D_s G)\,x\end{aligned}$$

Consequently

$$(D\,H)\,x \equiv ((D\,F)\,x)+(D\,G)\,x$$

Finally, then

$$(D\,(F)+G)\,x \equiv ((D\,F)+D\,G)\,x$$

or, in abbreviated form,

$$D\,(F)+G \equiv (D\,F)+D\,G \qquad (5.5)$$

In other words, the slope of the sum of two functions F and G is the sum of the slope functions of F and G individually.

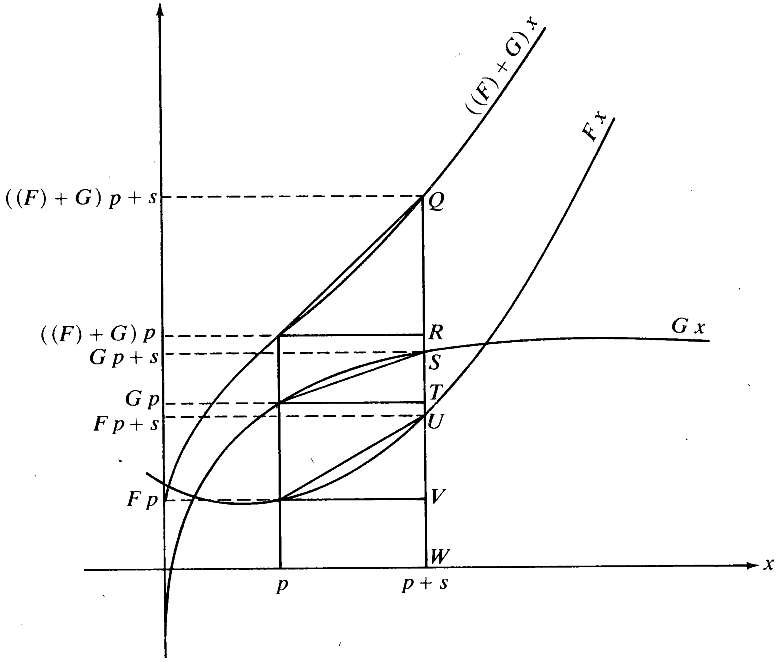


Figure 5.8 The slope of the sum of two functions

This result is illustrated geometrically in Figure 5.8. If $P\,Q$ represents the length of a line segment joining points P and Q, then

$$(D_s\,F)\,p \equiv \frac{U\,W - V\,W}{s} \equiv \frac{U\,V}{s}$$

$$(D_s\,G)\,p \equiv \frac{S\,W - T\,W}{s} \equiv \frac{S\,T}{s}$$

$$(D_s\,(F) + G)\,p \equiv \frac{Q\,W - R\,W}{s} \equiv \frac{Q\,R}{s}$$

But since $((F) + G)\,p \equiv (F\,p) + G\,p$, then

$$R\,W \equiv V\,W + T\,W$$

Similarly,

$$Q\,W \equiv U\,W + S\,W$$

Taking the difference of these two expressions and dividing by s yields the relation

$$(D_s\,(F) + G)\,p \equiv (D_s\,F)\,p + (D_s\,G)\,p$$

Since this is true for all values of s, then

$$(D\,(F) + G) \equiv (D\,F) + D\,G$$

It is easy to show that the rule for sums extends to three or more functions. For example, if

$$H \equiv (P) + (Q) + R$$

then

$$H \equiv (P) + ((Q) + R)$$

and the sum rule yields

$$D\,H \equiv (D\,P) + D\,((Q) + R)$$

But

$$D\,((Q) + R) \equiv (D\,Q) + D\,R$$

Hence

$$D\,H \equiv (D\,P) + (D\,Q) + D\,R$$

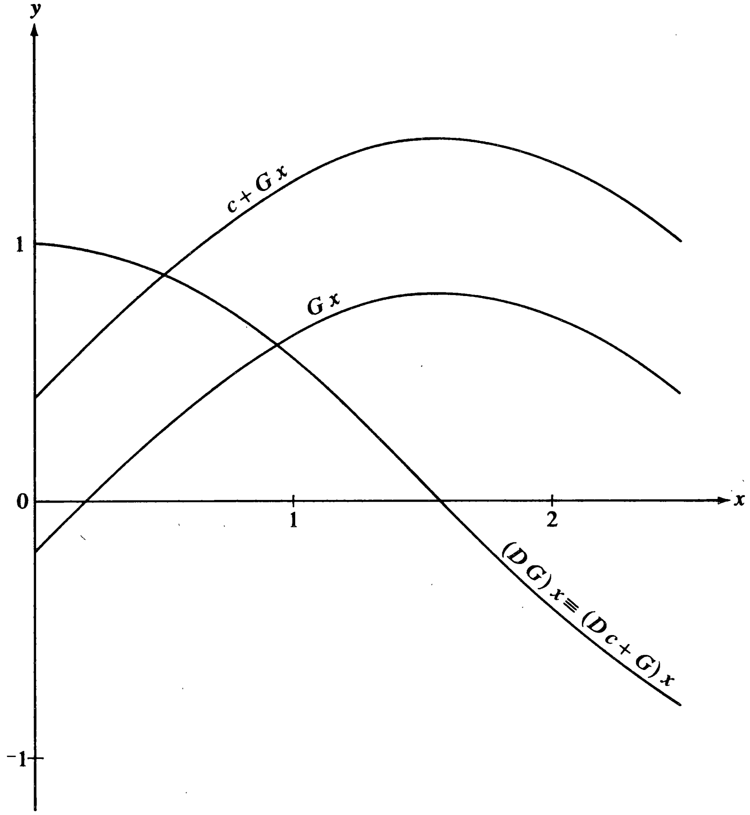


Figure 5.9 Functions having the same slope

If F is a constant function $F\,x \equiv c$, then $D\,F \equiv 0$, and the slope of the function $(F) + G \equiv c + G$ as given by Equation 5.5 is

$$(D\,c + G) \equiv (D\,F) + D\,G \equiv 0 + D\,G \equiv D\,G$$

In other words, the functions G and $c + G$ have the same slope, as illustrated by Figure 5.9. Conversely, any two functions having the same slope function can differ only by an additive constant. **(Do Exercise 5.9.)**

The Slope of the Product of Two Functions

The function $H\,x \equiv (F\,x) \times G\,x$ is called the *product* of $F\,x$ and $G\,x$. From Equation 5.1,

$$(D_s\,H)\,x \equiv \frac{((F\,x + s) \times G\,x + s) - (F\,x) \times G\,x}{s}$$

Regrouping the terms and adding and subtracting the term $(F\,x+s)$ $\times G\,x$ yields

$$(D_s H)\,x \equiv \frac{((F\,x+s)\times(G\,x+s)-G\,x)+(G\,x)\times(F\,x+s)-F\,x}{s}$$
$$\equiv ((F\,x+s)\times(D_s G)\,x)+(G\,x)\times(D_s F)\,x$$

For $s=0$, $(F\,x+s)\equiv F\,x$, and hence

$$(D\,H)\,x \equiv ((F\,x)\times(D\,G)\,x)+(G\,x)\times(D\,F)\,x$$

Since $H\equiv (F\,)\times G$, the abbreviated form of this result becomes

$$D\,(F\,)\times G \equiv ((F\,)\times D\,G)+(D\,F)\times G \qquad (5.6)$$

For example, if $F\equiv G\equiv (\quad)$, then $H\,x\equiv (F\,x)\times G\,x\equiv x\times x$, and $(D\,H)\,x\equiv (x\times 1)+(1\times x)\equiv 2\times x$. This agrees with the result previously obtained (Equation 5.4) for $x \ast 2$.

The result in Equation 5.4 can be checked further as follows. If $F\,x\equiv x$ and $G\,x\equiv x\ast 2$, and $H\,x\equiv x\ast 3$, then $H\,x\equiv (F\,x)\times G\,x$. Therefore

$$\begin{aligned}(D\,H)\,x &\equiv ((F\,x)\times(D\,G)\,x)+((D\,F)\,x)\times G\,x\\ &\equiv (x\times(2\times x))+1\times(x\ast 2)\\ &\equiv (2\times x\ast 2)+x\ast 2\equiv 3\times x\ast 2\end{aligned}$$

in agreement with Equation 5.4.

If $F\,x\equiv c$, then $D\,F\equiv 0$ and

$$\begin{aligned}(D\,(F\,)\times G)\,x &\equiv (c\times(D\,G)\,x)+0\times G\,x\\ &\equiv c\times(D\,G)\,x\end{aligned}$$

Hence

$$D\,c\times G\equiv c\times D\,G$$

In other words, multiplying a function by a constant c multiplies its slope function by c as well. In particular, for $c={}^{-}1$, it follows that $D\,({}^{-}1\times G)\equiv -D\,G$, and therefore $D\,(-G)\equiv -D\,G$. **(Do Exercises 5.10–5.11.)**

The Slope Function of the Polynomial

Since a polynomial is a *sum* of terms each of which is a *product* of a constant and a function of the form $x\ast n$, the slope of a polynomial can now be derived by a simple application of the results obtained thus far.

For example, if $\boldsymbol{c} = (2\ ,4\ ,3)$, then

$$H\,x \equiv \boldsymbol{c}\,\Pi\,x \equiv (2\times x\ast 0) + (4\times x\ast 1) + (3\times x\ast 2)$$

and

$$(D\,H)\,x \equiv (D\,2\times x\ast 0) + (D\,4\times x\ast 1) + (D\,3\times x\ast 2)$$

(applying Equation 5.5 for sums of functions). Applying the rule for multiplication by a constant (that is, $D\,c\times F \equiv c\times D\,F$) yields

$$(D\,H)\,x \equiv (2\times D\,x\ast 0) + (4\times D\,x\ast 1) + (3\times D\,x\ast 2)$$

Applying Equation 5.4 for powers yields

$$\begin{aligned}(D\,H)\,x &\equiv (2\times 0\times x\ast {}^{-}1) + (4\times 1\times x\ast 0) + (3\times 2\times x\ast 1)\\ &\equiv 0+4+6\times x \equiv (4\ ,6)\,\Pi\,x\end{aligned}$$

More generally, for any vector of coefficients $\boldsymbol{c}$,

$$\begin{aligned}\boldsymbol{c}\,\Pi\,x \equiv (\boldsymbol{c}_1\times x\ast 0) + (\boldsymbol{c}_2\times x\ast 1) + (\boldsymbol{c}_3\times x\ast 2)&\\ + (\boldsymbol{c}_4\times x\ast 3) + \ldots + \boldsymbol{c}_{\rho c}\times x\ast (\rho\,\boldsymbol{c}) - 1&\end{aligned}$$

and

$$\begin{aligned}(D\,\boldsymbol{c}\,\Pi)\,x &\equiv (0\times \boldsymbol{c}_1) + (1\times \boldsymbol{c}_2\times x\ast 0) + (2\times \boldsymbol{c}_3\times x\ast 1)\\ &\quad + (3\times \boldsymbol{c}_4\times x\ast 2) + \ldots + ((\rho\,\boldsymbol{c}) - 1)\times \boldsymbol{c}_{\rho c}\times x\ast (\rho\,\boldsymbol{c}) - 2\\ &\equiv \boldsymbol{d}\,\Pi\,x\end{aligned}$$

where $\boldsymbol{d}_1 \equiv 1\times \boldsymbol{c}_2$; $\boldsymbol{d}_2 \equiv 2\times \boldsymbol{c}_3$, and so on. In general, it is clear that $\boldsymbol{d}$ is obtained by deleting the first component from the vector $\boldsymbol{c}$ and then multiplying successive components by 1, 2, 3, ..., $(\rho\,\boldsymbol{c}) - 1$. Hence

$$\boldsymbol{d} \equiv (\iota\,(\rho\,\boldsymbol{c}) - 1)\times (1 < \iota\,\rho\,\boldsymbol{c})/\boldsymbol{c}$$

Finally,

$$(D\,\boldsymbol{c}\,\Pi)\,x \equiv ((\iota\,(\rho\,\boldsymbol{c}) - 1)\times (1 < \iota\,\rho\,\boldsymbol{c})/\,\boldsymbol{c})\,\Pi\,x$$

or

$$D\,\boldsymbol{c}\,\Pi \equiv ((\iota\,(\rho\,\boldsymbol{c}) - 1)\times (1 < \iota\,\rho\,\boldsymbol{c})/\,\boldsymbol{c})\,\Pi \qquad (5.7)$$

The rule for obtaining the slope function of any polynomial is therefore extremely simple: remove the first coefficient and multiply succeeding coefficients by 1, 2, 3, and so forth.

For example, if

$$F\,x \equiv 6 + (3 \times x) + (2 \times x^2) - x^4$$

then

$$F\,x \equiv (6\,,3\,,2\,,0\,,^{-}1)\,\Pi\,x$$

and

$$(D\,F)\,x \equiv (3\,,4\,,0\,,^{-}4)\,\Pi\,x$$
$$\equiv 3 + (4 \times x) - 4 \times x^3$$

(Do Exercises 5.12–5.13.)

EXAMPLE 1: The function graphed in Figure 5.5 has zero slope at the points (0, 0) and $\left(2, \frac{4}{3}\right)$. These points are clearly a *local minimum* and *local maximum* respectively.† For any elementary function the points of zero slope are rather easily determined, and the point where the function achieves its maximum can be easily selected from among them. The interest in determining the maximum can be illustrated by the following problem: Small squares, all of the same size, are to be cut from the corners of a one-foot square piece of sheet metal so that the remaining sheet can be folded up into a topless box. What size should the small squares be to yield a box of the largest possible volume? If the squares have sides of length x, then (as seen from Figure 5.10) the volume v is given by

$$v \equiv V\,x \equiv x \times (1 - 2 \times x) * 2$$
$$\equiv (0\,,1\,,^{-}4\,,4)\,\Pi\,x$$

The maximum value of the volume is found by determining the points at which the function $V\,x$ has zero slope. The slope function of $V\,x$ is given by

$$(D\,V)\,x \equiv ((1\,,2\,,3) \times (1\,,^{-}4\,,4))\,\Pi\,x$$
$$\equiv (1\,,^{-}8\,,12)\,\Pi\,x$$

Equating $(D\,V)\,x$ to zero and solving for x yields solutions $x \equiv \frac{1}{6}$ and $x \equiv \frac{1}{2}$. Evaluating $V\,x$ for these values of x gives $V\,1 \div 6 \equiv \frac{2}{27}$ and

---

†A *local minimum* is a point on the graph of a function which is lower than all *nearby* points. A local minimum is not necessarily a minimum. For example, in Figure 5.5 the point (0 , 0) is a local minimum of the function $F\,x$, but is not a minimum, since other points—such as (3.1 , $^{-}$1.089)—are lower.

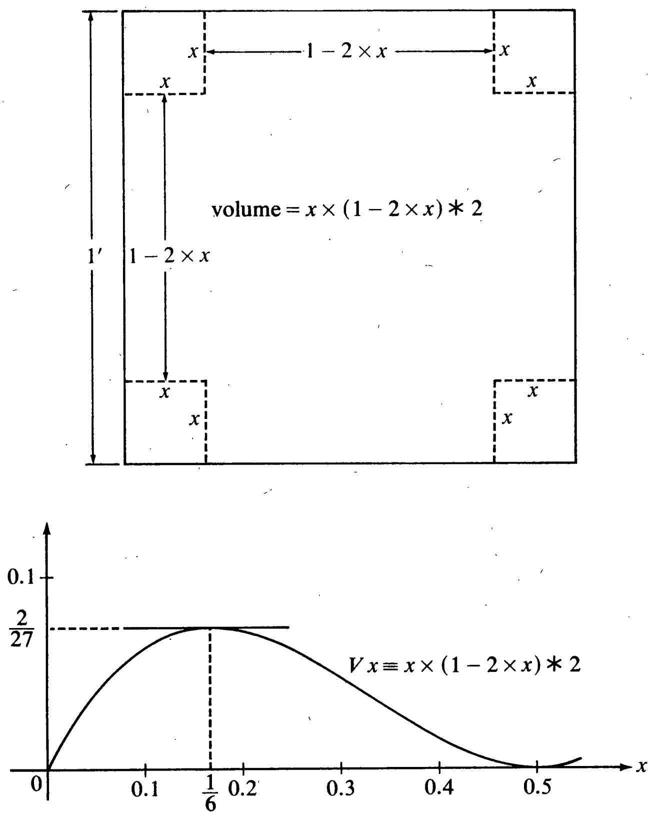


Figure 5.10 The maximum volume for a topless box

$V\,1 \div 2 \equiv 0$. Hence (as illustrated by the graph of $V\,x$ in Figure 5.10) the function attains a minimum at $(\frac{1}{2}, 0)$ and a maximum at $(\frac{1}{6}, \frac{2}{27})$; the maximum volume obtainable is $\frac{2}{27}$ cubic feet.

(Do Exercise 5.14.)

EXAMPLE 2: Figure 5.11 shows the function $P\,t$ which describes the position of a falling body (measured from its point of release at the top of a well at time $t = 0$) as a function of time. The function $V\,t$ is the slope of $P\,t$. But $V\,t$ is the limit (as s approaches zero) of the expression

$$V_s\,t \equiv \frac{(P\,t + s) - P\,t}{s}$$

which gives the average velocity during the time interval from t to $t + s$. Hence the function $V\,t$ gives the *velocity* at time t. Similarly, the function $A\ t$ is the slope of $V\ t$ and represents the *acceleration* at time t. In the present example it is seen to be a constant, as it should be in the case of free fall.

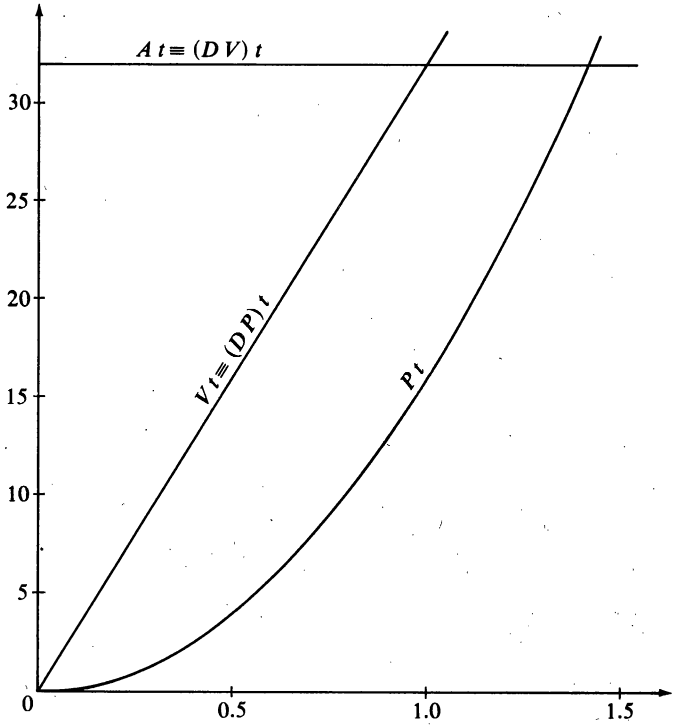


Figure 5.11 Position, velocity, and acceleration of a freely falling body as a function of time

It is instructive to consider the same functions in the opposite order. If only the acceleration is known, then a function having this slope can be determined. This function is $V\ t$. Likewise, a function having $V\ t$ as its slope can be determined; this is the function $P\ t$. Furthermore, if the acceleration $A\ t$ is equal to the constant value 32 (which is approximately true for free fall), then $V\ t$ must be equal to $c + 32 \times t$, where c is some constant. For, if $V\ t \equiv c + 32 \times t$, then $(D\ V)\ t \equiv 32$ as required. Moreover, if the velocity is zero when the body is dropped (that is, at time $t = 0$), then $V\ 0$ must be zero and therefore $c = 0$. Finally,

$$V\ t \equiv 32 \times t$$

Similarly, $P\,t$ must be equal to $d + 16 \times t^2$ for some constant d, since $(D\,P)\,t$ is then equal to $2 \times 16 \times t \equiv 32 \times t \equiv V\,t$ as required. Again, since the position at time $t = 0$ is zero, the constant d must be zero. Hence

$$P\,t \equiv 16 \times t^2$$

(Do Exercise 5.15.)

Some Interesting Functions

Certain natural phenomena are described by functions with the following characteristic: the slope of the function is proportional to the function. For example, since bacteria multiply by division, the population of a well-fed colony of bacteria continually increases at a rate proportional to the population. Thus, if $P\,t$ gives the population as a function of time t, then

$$(D\,P)\,t \equiv r \times P\,t$$

where r is a constant of proportionality.

Other examples of this type of function abound. A tree grows at a rate (approximately) proportional to its present size; the rate of discharge from a hole at the bottom of a tank of water is proportional to the pressure and hence to the amount of water in the tank at each instant; the rate of discharge from an electrical condenser connected through a resistor is at every instant proportional to the voltage and hence to the amount of electrical charge remaining.

Is it possible to find a polynomial whose slope function is equal to the polynomial? Clearly not, for the slope function is a polynomial of degree one less than the original polynomial. However, a polynomial that approximates this behavior can be found by simply choosing the coefficients so that the coefficients of the slope polynomial agree for all save the last.

If $\boldsymbol{c} \equiv 1 \div !\,0\,,\iota\,n$, then

$$\boldsymbol{c} \equiv \left(\frac{1}{!\,0},\frac{1}{!\,1},\frac{1}{!\,2},\frac{1}{!\,3},\ldots,\frac{1}{!\,n-1},\frac{1}{!\,n}\right)$$

and

$$\boldsymbol{c}\,\Pi\,x \equiv 1 + x + \frac{x^2}{!\,2} + \frac{x^3}{!\,3} + \frac{x^4}{!\,4} + \ldots + \frac{x^n}{!\,n}$$

is clearly a polynomial satisfying the requirements. For the slope function is $\boldsymbol{d}\,\Pi\, x$, where

$$\begin{aligned}\boldsymbol{d} &\equiv \left(\frac{1}{!\,1}, \frac{1}{!\,2}, \frac{1}{!\,3}, \ldots, \frac{1}{!\,n}\right) \times (1, 2, 3, 4, \ldots, n)\\ &\equiv \left(1, 1, \frac{1}{!\,2}, \ldots, \frac{1}{!\,n-1}\right)\end{aligned}$$

Hence

$$\boldsymbol{c} \equiv \boldsymbol{d}, \boldsymbol{c}_{\rho \boldsymbol{c}}$$

and

$$\begin{aligned}\boldsymbol{d}\,\Pi\, x &\equiv 1 + x + \frac{x^2}{!\,2} + \frac{x^3}{!\,3} + \ldots + \frac{x^{n-1}}{!\,n-1}\\ &\equiv (\boldsymbol{c}\,\Pi\, x) - \frac{x^n}{!\,n}\end{aligned}$$

The slope of $\boldsymbol{c}\,\Pi\, x$ therefore differs from $\boldsymbol{c}\,\Pi\, x$ only by the term $(x \ast n) \div !\, n$, where $n \equiv (\rho\, \boldsymbol{c}) - 1$. Because of the factor $!\, n$ in the denominator this difference can be made as small as desired by choosing n sufficiently large. For n sufficiently large, the value of the polynomial $\boldsymbol{c}\,\Pi\, x$ therefore approaches a limiting value which will be denoted by $\ast\, x$.† Thus

$$\begin{aligned}\ast\, x &\equiv (1 \div !\, 0, \iota\, n)\, \Pi\, x\\ &\equiv 1 + x + \frac{x^2}{!\,2} + \frac{x^3}{!\,3} + \frac{x^4}{!\,4} + \ldots \qquad (5.8)\end{aligned}$$

and $\ast\, x$ has the desired property, namely,

$$(D \ast)\, x \equiv \ast\, x \qquad (5.9)$$

---

†This use of the symbol $\ast$ for a monadic function does not conflict with the earlier use for the dyadic function of exponentiation. In Chapter 8 it will be shown that the monadic function $\ast$ is, in fact, the special case of the dyadic function $e \ast x$, where $e \equiv \ast\, 1 \equiv 2.71828\ldots$

Although it has an infinite number of terms, the polynomial $\ast x$ can be computed to any desired degree of accuracy because the coefficients of the higher-order terms decrease so rapidly. For example

$$\begin{aligned}\ast 1 &\equiv 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040}+\dots\\ &\equiv 2.71828\dots\end{aligned}$$

as can be verified by simple but tedious arithmetic. The general behavior of $\ast x$ can be determined by evaluating and plotting the function for a number of values of x as shown in Figure 5.4.

The slope of the function

$$(\ast\, r\times)\, x \equiv \ast\, r\times x \equiv 1+(r\times x)+\frac{(r\times x)^2}{!\,2}+\frac{(r\times x)^3}{!\,3}+\dots$$

can be obtained by applying Equation 5.7:

$$(D\ast r\times)\, x \equiv r+(r^2\times x)+\frac{r^3\times x^2}{!\,2}+\frac{r^4\times x^3}{!\,3}+\dots$$

Hence

$$(D\ast\, r\times)\, x \equiv r\times(\ast\, r\times x) \qquad (5.10)$$

In other words, $(\ast\, r\times)$ is a function whose slope is proportional to itself, the constant of proportionality being r. It is therefore a function of the form suggested at the beginning of this section.

(Do Exercises 5.16–5.19.)

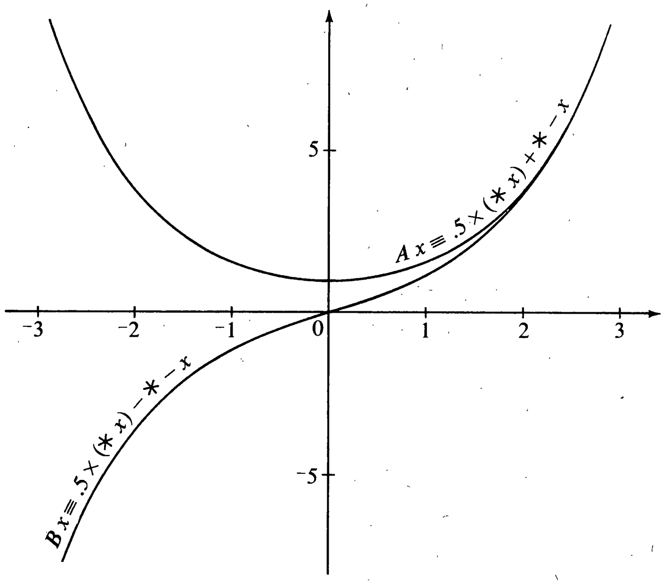
The average of the polynomials $\ast x$ and $\ast -x$ yields another useful function, which will be called $A\,x$. Specifically

$$\begin{aligned}A\,x &\equiv .5\times(\ast\, x)+\ast\,-x\\ &\equiv 1+\frac{x^2}{!\,2}+\frac{x^4}{!\,4}+\frac{x^6}{!\,6}+\dots\end{aligned} \qquad (5.11)$$

Since $A\,x$ contains only even powers of x it is an even function of x, that is

$$A - x \equiv A\,x$$

The function is graphed in Figure 5.12.



x	$\ast x$	$A\,x$	$B\,x$
¯3	0.050	10.068	10.018
¯2	0.135	3.762	3.627
¯1	0.368	1.543	1.175
0	1.000	1.000	0.000
1	2.718	1.543	¯1.175
2	7.389	3.762	¯3.627
3	20.086	10.068	¯10.018

Figure 5.12 The functions $A\,x$ and $B\,x$

One-half the *difference* of $\ast\, x$ and $\ast - x$ yields a third function, which will be called $B\,x$; thus

$$\begin{aligned} B\,x &\equiv .5 \times (\ast\, x) - \ast - x \\ &\equiv x + \frac{x^3}{!\,3} + \frac{x^5}{!\,5} + \frac{x^7}{!\,7} + \dots \end{aligned} \qquad (5.12)$$

Since $B\,x$ contains only odd powers of x, it is an odd function of x, that is, $B - x \equiv - B\,x$. It is also graphed in Figure 5.12.

From the definitions of A and B it is clear that

$$(A) + B \equiv \ast \text{ and } (A) - B \equiv \ast -$$

Since

$$A\,x \equiv \left(1\,,0\,,\frac{1}{!\,2},0\,,\frac{1}{!\,4},0\,,\frac{1}{!\,6},0\,,\ldots\right)\Pi\,x$$

and

$$B\,x \equiv \left(0\,,1\,,0\,,\frac{1}{!\,3},0\,,\frac{1}{!\,5},0\,,\frac{1}{!\,7},\ldots\right)\Pi\,x$$

it is easily shown (by applying Equation 5.7) that

$$\left.\begin{aligned} D\,A &\equiv B \\ D\,B &\equiv A \end{aligned}\right\} \qquad (5.13)$$

Since a slope function $D\,F$ is itself a function, it is possible to determine *its* slope function $D\,(D\,F)$, the slope of that function $D\,(D\,(D\,F))$, and so on. Just as the slope function $D\,F$ is sometimes called the *derivative* of F, so the function $D\,D\,F$ is called the *second derivative* of F; $D\,D\,D\,F$ is called the *third derivative* of F, and so on.

For the function A it is clear that

$$D\,(D\,A) \equiv A$$

Hence A is a function which is equal to its own *second* derivative. Clearly B is a similar function, since $D\,(D\,B) \equiv D\,A \equiv B$.

The graph of the function $A\,x$ is the form assumed by a cable suspended between two supports.

(Do Exercises 5.20–5.28.)

Two more interesting functions can be obtained by reversing the signs of alternate nonzero terms of $A\,x$ and $B\,x$. They will be denoted by $C\,x$ and $S\,x$ and defined as follows:

$$\left.\begin{aligned} C\,x &\equiv \left(1\,,0\,,\frac{-1}{!\,2},0\,,\frac{1}{!\,4},0\,,\frac{-1}{!\,6},\ldots\right)\Pi\,x \\ S\,x &\equiv \left(0\,,1\,,0\,,\frac{-1}{!\,3},0\,,\frac{1}{!\,5},0\,,\ldots\right)\Pi\,x \end{aligned}\right\} \qquad (5.14)$$

It is easily shown that

$$\left.\begin{aligned} D\,C &\equiv -\,S \\ D\,S &\equiv C \end{aligned}\right\} \qquad (5.15)$$

Consequently

$$\left.\begin{array}{l} D\,D\,C \equiv D\,(-S) \equiv -D\,S \equiv -C \\ D\,D\,S \equiv D\,C \equiv -S \end{array}\right\} \qquad (5.16)$$

Thus $C\,x$ is a function whose second derivative is equal to $-\,C\,x$. The function $S\,x$ behaves similarly.

The functions C and S have a curious property, which is easily derived from the behavior of their slopes, namely

$$((C) \times C) + (S) \times S \equiv 1 \qquad (5.17)$$

For if $H \equiv ((C) \times C) + (S) \times S$, then

$$\begin{aligned} D\,H &\equiv (((C) \times D\,C) + (C) \times D\,C) + ((S) \times D\,S) + (S) \times D\,S \\ &\equiv 2 \times ((C) \times D\,C) + (S) \times D\,S \\ &\equiv 2 \times ((C) \times -S) + (S) \times C \\ &\equiv 0 \end{aligned}$$

Since the slope function of H is zero, the function $H\,x$ must be a constant. The value of this constant can be determined easily by evaluating $H\,x$ for $x \equiv 0$. Since $C\,0 \equiv 1$, and $S\,0 \equiv 0$, then $H\,0 \equiv (1 \times 1) + 0 \times 0 \equiv 1$. This result can be tested by computing the first few terms of the indicated polynomial.

It is clear from Equation 5.17 that the value of $C\,x$ must lie between $^-1$ and 1 and that the same is true of $S\,x$. Because $C\,0 \equiv 1$ is the slope of $S\,x$ at $x \equiv 0$, $S\,x$ is rising from its value of 0 at $x \equiv 0$. When $S\,x$ is increasing $C\,x$ must (according to Equation 5.17) be decreasing; it continues to decrease until its value reaches $^-1$, at which point it again begins to increase until it reaches 1.

It is therefore not surprising that the functions $C\,x$ and $S\,x$ describe oscillations such as that of a weight suspended on a spring. If $P\,t$ describes the position of a weight suspended on a spring as a function of time t, then, as remarked in connection with Figure 5.11, $(D\,P)\,t$ is the velocity of the weight and $(D\,D\,P)\,t$ is the acceleration. Because the weight is supported by a spring, the acceleration applied to the mass is proportional to the displacement from the equilibrium position and is oppositely directed, that is, $D\,D\,P \equiv -\,P$. Thus the function P has the same property as that shown by Equation 5.16 for the functions C and S. These matters will be treated further in Chapter 6.

(Do Exercises 5.29–5.31.)

Approximating Polynomials of Unlimited Degree

The polynomial $Pf \equiv 1+f+f^2+f^3+\ldots$ of unlimited degree can be evaluated precisely for nonnegative values of f less than 1:

$$Pf \equiv \frac{1}{1-f} \tag{5.18}$$

For if $F\,n$ denotes the sum of the first n terms of the polynomial, then

$$F\,n \equiv 1+f+f^2+f^3+\ldots+f^{n-1}$$

and

$$f \times F\,n \equiv f+f^2+f^3+f^4+\ldots+f^n$$

Consequently $(F\,n) - f \times F\,n \equiv 1-f^n$, and hence

$$F\,n \equiv \frac{1-f^n}{1-f}$$

As n becomes large, the value of $F\,n$ approaches the value of Pf and, since $0 \leq f < 1$, the term f^n approaches 0, yielding Equation 5.18. For example:

$$P\,.9 \equiv 1+.9+.9^2+.9^3+\ldots \equiv \frac{1}{1-.9} \equiv 10$$

(Do Exercise 5.32.)

In evaluating any polynomial of unlimited degree (such as $* \, x$) it seems obvious that any desired degree of accuracy can be attained by taking a sufficiently large number of terms, provided that the later terms continually decrease in size. In particular, it would appear that the maximum extent of the error is indicated by the magnitude of the first neglected term. Consider, however, the evaluation of the polynomial

$$Pf \equiv 1+f+f^2+f^3+f^4+\ldots$$

for $f = 0.9$. The eighth term, f^7, has the value 0.4782969, yet the sum of the preceding seven terms is only 5.217031 and differs from the true value of $P\,0.9$ (which is 10) by 4.782969, an amount greatly exceeding the first neglected term.

It is evident that the first neglected term is not a sufficiently accurate measure of the potential error and that a better criterion is needed. Finding a suitable upper bound for the sum of all the neglected terms is a difficult problem that has received much attention in mathematics, in the theory of convergence of series. For the rather simple polynomials of interest in elementary functions, however, a simple criterion is easily obtained.

Let $\boldsymbol{t}$ be the vector of terms of the polynomial to be evaluated, let $\boldsymbol{s}$ be the vector of the first i terms actually summed, and let $\boldsymbol{r}$ be the vector of the remainder terms (that is, those not summed). Then $\boldsymbol{t} \equiv \boldsymbol{s}\,,\boldsymbol{r}$; the true sum is $+/\,\boldsymbol{t} \equiv (+/\,\boldsymbol{s}) + (+/\,\boldsymbol{r})$; the calculated sum is $+/\,\boldsymbol{s}$; and the error due to the remainder is $+/\,\boldsymbol{r}$.

What is needed is an upper bound on the absolute value of the error (that is, $|+/\,\boldsymbol{r}$) rather than on the error itself. For example, an error of $^-678$ is *less than* a tolerance $t = 0.001$, but is clearly unacceptable because the magnitude of the error (that is, $|\,^-678 \equiv 678$) is too large. Since $(|+/\,\boldsymbol{r}) \leq +/\,|\,\boldsymbol{r}$, the required upper bound can be estimated by summing the components of the vector $|\,\boldsymbol{r}$. The absolute value of the vector $\boldsymbol{r}$ will now be denoted by $\boldsymbol{a}$, that is, $\boldsymbol{a} \equiv |\,\boldsymbol{r}$.

In general, $+/\,\boldsymbol{a}$ cannot be evaluated directly, but it is frequently possible to find a vector $\boldsymbol{d}$ which *can* be summed and which *dominates* $\boldsymbol{a}$, that is, $\boldsymbol{d}_j \geq \boldsymbol{a}_j$ for all j. Clearly, $(+/\,\boldsymbol{d}) \geq (+/\,\boldsymbol{a}) \equiv (+/\,|\,\boldsymbol{r}) \geq (|+/\,\boldsymbol{r})$, and therefore the sum over $\boldsymbol{d}$ provides an upper bound on the error incurred by neglecting the remainder $+/\,\boldsymbol{r}$.

If $\boldsymbol{a}_{j+1} \leq \boldsymbol{a}_j \times f$ for all values of j, then $\boldsymbol{a}_k \leq \boldsymbol{a}_1 \times f \ast k - 1$. Therefore the vector

$$\boldsymbol{d} \equiv \boldsymbol{a}_1 \times (1\,, f\,, f^2\,, f^3\,, \ldots)$$

dominates $\boldsymbol{a}$. If the factor f is less than 1, then from Equation 5.18

$$+/\,\boldsymbol{d} \equiv \boldsymbol{a}_1 \times \frac{1}{1-f}$$

If the ratio between successive terms of $\boldsymbol{a}$ is not increasing (that is, $\dfrac{\boldsymbol{a}_{j+1}}{\boldsymbol{a}_j} \leq \dfrac{\boldsymbol{a}_j}{\boldsymbol{a}_{j-1}}$), then $\boldsymbol{d}$ will dominate $\boldsymbol{a}$ if f is chosen equal to $\dfrac{\boldsymbol{a}_2}{\boldsymbol{a}_1}$. Hence

$$(|+/\,\boldsymbol{r}) \leq (+/\,\boldsymbol{a}) \leq (+/\,\boldsymbol{d}) \equiv \frac{\boldsymbol{a}_1}{1 - \boldsymbol{a}_2 \div \boldsymbol{a}}$$

Since $\boldsymbol{t} \equiv \boldsymbol{s}\,,\boldsymbol{r}$ and $(\rho\,\boldsymbol{s}) \equiv i$ and $\boldsymbol{a} \equiv |\,\boldsymbol{r}$, then $\boldsymbol{a}_1 \equiv |\,\boldsymbol{t}_{i+1}$ and $\boldsymbol{a}_2 \equiv |\,\boldsymbol{t}_{i+2}$. Therefore, in terms of the original polynomial $+/\,\boldsymbol{t}$, the error bound on $+/\,\boldsymbol{r}$ is given by

$$(|+/\,\boldsymbol{r}) \leq \frac{|\,\boldsymbol{t}_{i+1}}{1 - (|\,\boldsymbol{t}_{i+2}) \div |\,\boldsymbol{t}_{i+1}} \equiv \frac{\boldsymbol{t}_{i+1} \ast 2}{(|\,\boldsymbol{t}_{i+1}) - |\,\boldsymbol{t}_{i+2}} \qquad (5.19)$$

In summary, the true sum of the terms $\boldsymbol{t}$ of a polynomial will differ from the sum of the first i components of $\boldsymbol{t}$ by an amount not exceeding the expression for $|+/\boldsymbol{r}$ in Equation 5.19, provided that for all $j>i$ the ratios $(|\boldsymbol{t}_{j+1})\div|\boldsymbol{t}_j$ do not exceed the ratio $(|\boldsymbol{t}_{i+2})\div|\boldsymbol{t}_{i+1}$, and that the ratio $(|\boldsymbol{t}_{i+2})\div|\boldsymbol{t}_{i+1}$ is less than 1.

For example, since

$$* 2 \equiv 1+2+\frac{4}{!2}+\frac{8}{!3}+\frac{16}{!4}+\frac{32}{!5}+\frac{64}{!6}+\ldots$$

then

$$* 2 \equiv +/\boldsymbol{t}\text{, where}$$

$$\boldsymbol{t} \equiv \left(1\ ,2\ ,\frac{4}{!2},\frac{8}{!3},\frac{16}{!4},\frac{32}{!5},\frac{64}{!6},\ldots\right)$$

Moreover, for $i=6$, the ratio $(|\boldsymbol{t}_{j+1})\div|\boldsymbol{t}_j$ is less than unity for $j>i$, and Equation 5.19 can therefore be applied to give the error bound $+/\boldsymbol{d}$. Therefore $+/\boldsymbol{d} \equiv \dfrac{\frac{64}{!6}}{1-\left(\frac{128}{!7}\right)\div\frac{64}{!6}} \equiv \frac{64}{!6}\div\frac{5}{7} \equiv 0.1244\ldots$ Hence the sum of the first six terms of $* 2$ gives an approximation of $* 2$ to within a possible error of 0.1244...

The polynomial

$$* x \equiv 1+x+\frac{x^2}{!2}+\frac{x^3}{!3}+\ldots$$

can be treated in a similar manner for any specified value of the argument x. For, since the ratio $|\boldsymbol{t}_{k+2}\div\boldsymbol{t}_{k+1}$ is equal to $\frac{|x}{k+1}$, all the required conditions are satisfied by choosing $i\geq x$. Similar arguments apply to the functions A, B, C, and S.

(Do Exercises 5.33–5.34.)

Applications of the Slope Function

Perhaps the most important application of the slope function is in the study of functions such as the circular functions and the exponential function. This use of the slope will occur repeatedly in later chapters. Applications to be treated here are the graphing of curves and the determination of areas and volumes enclosed by certain curves.

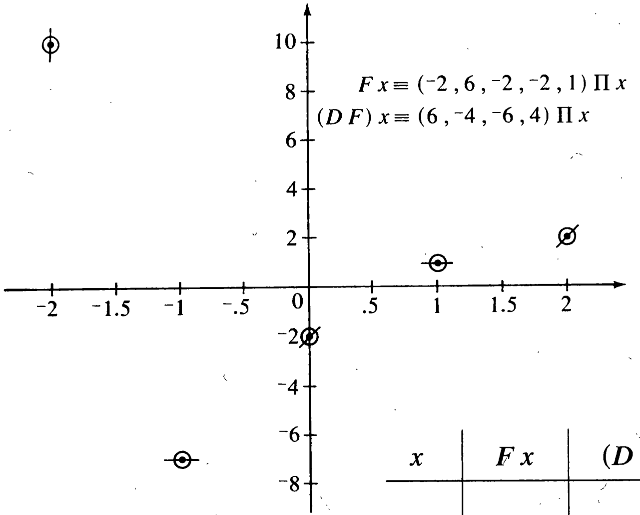
Curve plotting. In making an accurate graph of a function, it is helpful to know the slope of the curve at each point plotted and to indicate this slope by drawing a short segment of the tangent line at each point. For example, if

$$F\,x \equiv (^{-}2\,,6\,,^{-}2\,,^{-}2\,,1)\;\Pi\,x$$

then

$$(D\,F)\,x \equiv (6\,,^{-}4\,,^{-}6\,,4)\;\Pi\,x$$

Figure 5.13 shows a tabulation of both these functions for a number of values and a plot of the tangent for each of the tabulated points.



x	$F\,x$	$(D\,F)\,x$
$^{-}2$	10	$^{-}42$
$^{-}1$	$^{-}7$	0
0	$^{-}2$	6
1	1	0
2	2	6

Figure 5.13 The use of tangents in graphing a function

It is clear from Figure 5.13 that the tangents give a much better picture of the curve than the points alone would. In particular, the points $(^{-}1\,,F\,^{-}1)$ and $(1\,,F\,1)$ are points of zero slope and are a local minimum and a local maximum respectively. Moreover the third zero of $(D\,F)\,x$ is easily found to be at $x = 1.5$; it is a second local minimum of $F\,x$ with a value of $\frac{13}{16}$.

The area under a curve. The area of the shaded portion of Figure 5.14 enclosed by the y-axis, the x-axis, the curve $F\ x$, and the line $x = t$ is clearly a function of t. It will be denoted by $G\ t$. If t is increased by a small amount s, the added area is approximately $s \times F\ t$ and therefore the slope of the function $G\ t$ is approximately $F\ t$. That is,

$$(D_s G)\ t \equiv \frac{((G\ t) + s \times F\ t) - G\ t}{s} \equiv F\ t \quad \text{(approximately)}$$

If, as will be shown,

$$(D\ G)\ t \equiv F\ t \tag{5.20}$$

exactly, then the area can be determined simply by finding the function G whose slope is F. If F is a polynomial, this is easily done by reversing the procedure described by Equation 5.7.

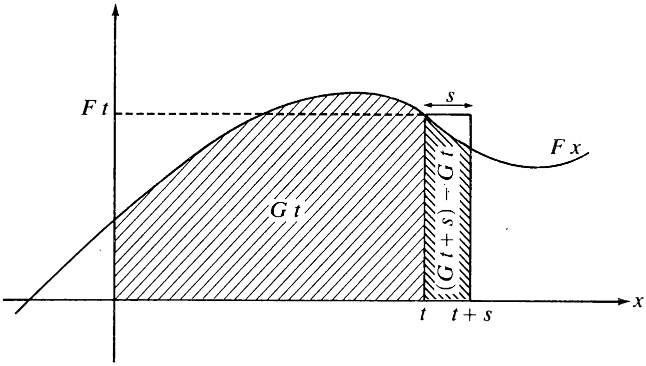


Figure 5.14 The area under a curve $F\ x$

For example, if $F\ x$ is the parabola

$$F\ x \equiv 9 - 4 \times x^2 \equiv (9\ ,0\ ,{}^{-}4)\ \Pi\ x$$

shown in Figure 5.15, then $G\ t$ is the polynomial whose coefficients are obtained from those of F by *dividing* successive components by 1, 2, 3, and then appending a leading component c. Thus

$$G\ t \equiv \left(c\ ,9\ ,0\ ,-\frac{4}{3}\right)\Pi\ t$$

Any value of c will satisfy Equation 5.20, and the value of c can therefore be chosen so that the area $G\ t$ is 0 for $t \equiv 0$, as required. Hence

$$0 \equiv G\ 0 \equiv \left(c\ ,9\ ,0\ ,-\frac{4}{3}\right)\Pi\ 0 \equiv c$$

and therefore $c \equiv 0$. Finally

$$G\ t \equiv \left(0\ ,9\ ,0\ ,-\frac{4}{3}\right)\Pi\ t \equiv (9 \times t) - \frac{4}{3} \times t^3$$

More generally, the area enclosed by the curve $F\ x$, the x-axis, and the lines $x = a$ and $x = t$ can be determined similarly. The only difference arises in the evaluation of the constant c. If this area is denoted by $H\ t$, then $H\ a$ must be zero and therefore for the curve $F\ x$ of the preceding example

$$0 \equiv H\ a \equiv \left(c\ ,9\ ,0\ ,-\frac{4}{3}\right)\Pi\ a$$

Therefore

$$0 \equiv c + (9 \times a) - \frac{4}{3} \times a^3$$

$$c \equiv \left(\frac{4}{3} \times a^3\right) - 9 \times a$$

and

$$H\ t \equiv \left(\left(\frac{4}{3} \times a^3\right) - 9 \times a\right) + (9 \times t) - \frac{4 \times t^3}{3}$$

For example, if $a = \frac{1}{2}$, then $c = -\frac{13}{3}$ and

$$H\ t \equiv \left(\left(-\frac{13}{3}\right),9\ ,0\ ,-\frac{4}{3}\right)\Pi\ t$$

The relation between the functions $G\ t$ and $H\ t$ is shown by their graphs in Figure 5.15; $H\ t$ is obtained by moving $G\ t$ vertically until $H\ a \equiv 0$.

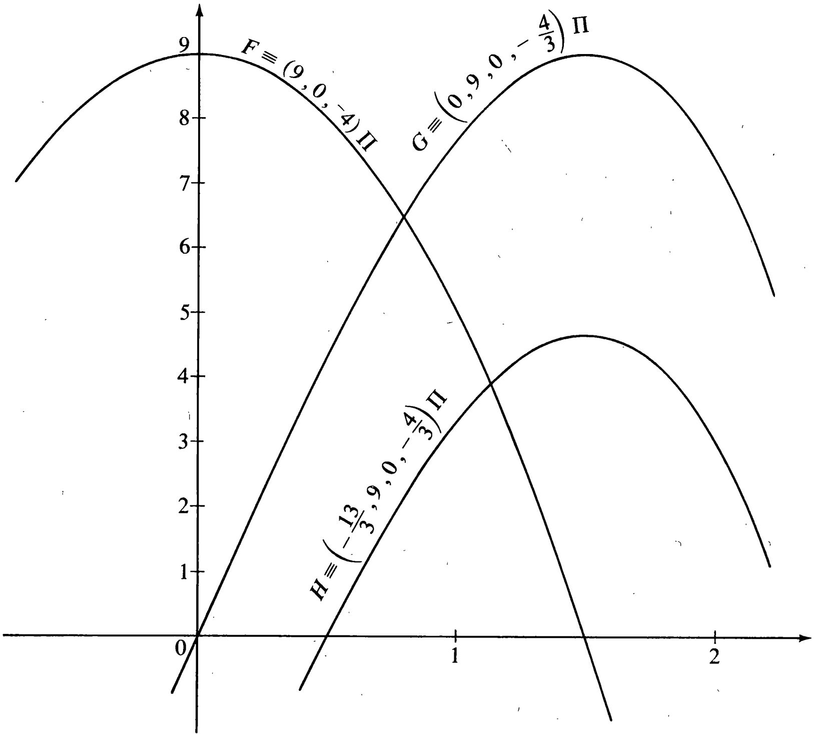


Figure 5.15 The area under $F \equiv (9\,, 0\,, {}^{-}4)\,\Pi$

This method of determining the area under a curve can be tested by applying it to the function

$$F\,x \equiv (p\,, q)\ \Pi\ x \equiv p + q \times x$$

whose area (Figure 5.16) is known from geometry. In this case

$$G\,t \equiv \left(c\,, p\,, \frac{q}{2}\right)\Pi\ t$$

and c is evaluated from the relation

$$G\,a \equiv \left(c\,, p\,, \frac{q}{2}\right)\Pi\ a \equiv 0$$

Hence $c \equiv -\left((p \times a) + \frac{q \times a^2}{2}\right)$, and finally

$$\begin{aligned} G\,t &\equiv \left(-\,(p \times a) + \frac{q \times a^2}{2}\right) + (p \times t) + \frac{q \times t^2}{2} \\ &\equiv (t-a) \times \left(p + \frac{q}{2} \times (t+a)\right) \\ &\equiv (t-a) \times .5 \times (p + q \times a) + p + q \times t \\ &\equiv (t-a) \times .5 \times (F\,a) + F\,t \end{aligned}$$

which is the appropriate expression for the area of a trapezoid.

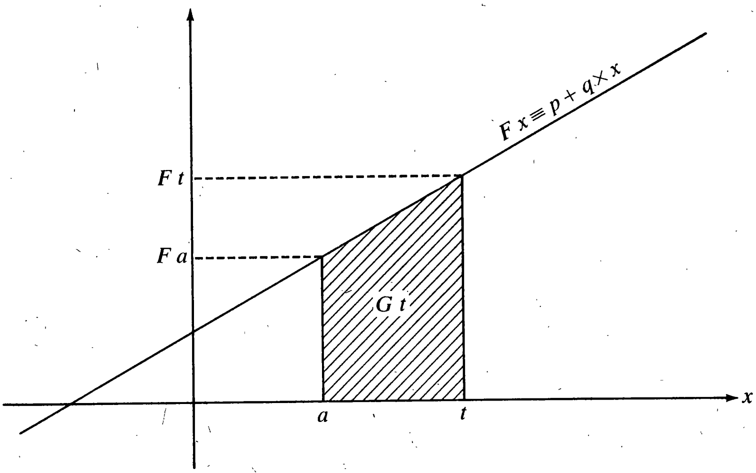


Figure 5.16 The area of a trapezoid

According to Equation 5.20, the area $G\,t$ enclosed by the curve $F\,x$, the x-axis, the line $x=0$, and the line $x=t$ satisfies the relation

$$(D\,G)\;t \equiv F\,t$$

To prove this, reconsider Figure 5.14. Clearly

$$((G\,t) + s \times F\,t) \geq (G\,t + s) \geq ((G\,t) + s \times F\,t + s)$$

Therefore

$$F\,t \geq \frac{(G\,t+s)-G\,t}{s} \geq F\,t+s$$

and

$$F\,t \geq (D_s G)\,t \geq F\,t+s$$

As s approaches zero, $(D_s G)\,t$ approaches $(D\,G)\,t$ and $F\,t+s$ approaches $F\,t$. Therefore $F\,t \geq (D\,G)\,t \geq F\,t$, and $(D\,G)\,t$ must equal $F\,t$. If the function $F\,t$ is increasing at the point t, the foregoing inequalities are all reversed, but the conclusion is obviously the same.

(Do Exercises 5.35–5.37.)

Volumes. Consider a solid such as the one shown in Figure 5.17, whose cross-sectional area in all planes perpendicular to the x-axis is some known function W of the distance x from the origin. Then the volume $V\,t$ cut off by a plane at $x = t$ can be found by methods analogous to those used for determining areas. From Figure 5.17 it appears that

$$(D\,V)\,t \equiv W\,t \qquad (5.21)$$

where $W\,t$ is the cross-sectional area of the plane at distance t from the origin. If $W\,t$ is a polynomial, the function $V\,t$ can be determined. Moreover the proof of Equation 5.21 follows the proof of Equation 5.20.

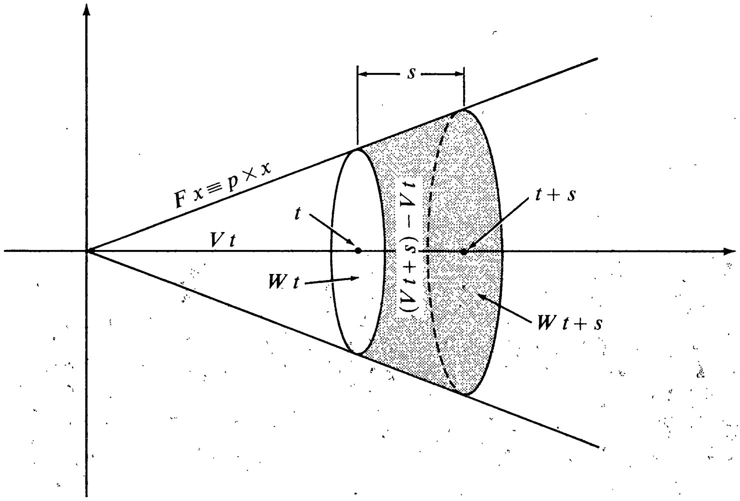


Figure 5.17 The volume of a cone

For example, in the cone of Figure 5.17,

$$\begin{aligned} W\,t &\equiv \pi \times (p \times t)^2 \\ &\equiv (\pi \times p^2) \times (0\,,0\,,1)\ \Pi\ t \end{aligned}$$

Therefore

$$V\,t \equiv (\pi \times p^2) \times \left(c\,,0\,,0\,,\frac{1}{3}\right) \Pi\ t$$

Again c can be evaluated by noting that $V\,0 = 0$ and therefore $c = 0$. Hence

$$\begin{aligned} V\,t &\equiv (\pi \times p^2) \times \left(0\,,0\,,0\,,\frac{1}{3}\right) \Pi\ t \\ &\equiv \frac{(\pi \times p^2)}{3} \times t^3 \end{aligned}$$

Since the area of the base of this cone is given by

$$b \equiv \pi \times (p \times t)^2$$

$V\,t$ can be reexpressed in the more familiar form

$$V\,t \equiv \frac{b \times t}{3}$$

For the volume U of the frustum of the cone bounded by the planes at distances a and t from the origin, the constant c can be evaluated accordingly. Thus

$$U\,t \equiv (\pi \times p^2) \times \left(c\,,0\,,0\,,\frac{1}{3}\right) \Pi\ t \text{ and } U\,a \equiv 0$$

Hence $c = \dfrac{-\,a^3}{3}$ and

$$U\,t \equiv \left(\frac{\pi \times p^2}{3}\right) \times (t^3 - a^3)$$

The solid cone of Figure 5.17 can be conceived as being generated by revolving the curve $F\,x \equiv p \times x$ about the x-axis. The volume of any solid so generated is called a *volume of revolution* generated by $F\,x$. Figure 5.18 shows the volume of revolution generated by the function $F\,x \equiv \sqrt{}x$. The cross-sectional area at distance t from the origin is clearly given by

$$W\,t \equiv \pi \times (\sqrt{}\ t)^2 \equiv \pi \times t \equiv \pi \times (0\,,1)\ \Pi\ t$$

Hence

$$V\,t \equiv \pi \times \left(c\,,0\,,\frac{1}{2}\right) \Pi\; t$$

and again

$$c = 0$$

Therefore

$$V\,t \equiv \pi \times \left(0\,,0\,,\frac{1}{2}\right) \Pi\; t \equiv .5 \times \pi \times t^2$$

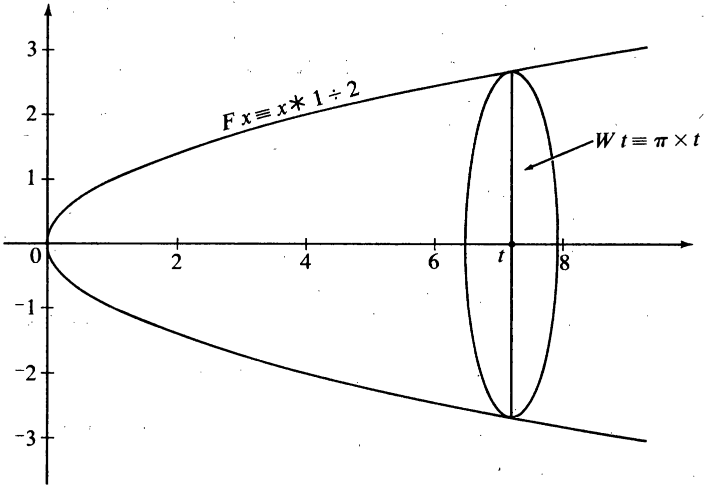


Figure 5.18 Volume of revolution of the square-root function

The foregoing methods for determining the areas and volumes enclosed by curves succeed only if the slope function obtained is one for which it is possible to determine another function having that slope. At present this means that the slope function must be a polynomial, and the method therefore fails in some seemingly simple cases such as determining the area of a circle (although the volume of a sphere can be determined). Later chapters will extend the set of functions which can be treated.

(Do Exercises 5.38–5.42.)

Exercises

NOTE: If a computer is available, it may be well to perform some of the indicated calculations by writing and executing programs.

5.1 (a) Draw tangents to the graph of the function $G\,x$ of Figure 5.3 at $x = {}^{-}1, \frac{{}^{-}1}{3}, 0, \frac{1}{3}, \frac{1}{2}$, and 1.

(b) Measure the slopes of these tangents, and use the measurements to make a table showing these slopes as a function of x for the six indicated points.

(c) Add a third column to the table of part (b) to show the values of $H\,x \equiv 1 + (2 \times x) - 3 \times x^2$ at the six points, and compare with the values obtained for the slopes of the tangents.

5.2 Repeat Exercise 5.1 for the graph of $E\,x$ of Figure 5.4 for $x = {}^{-}1.5, {}^{-}.5, .5$, and 1.5, making the comparison with the value of $E\,x$ at each of the points.

5.3 (a) Graph the function $F\,x \equiv x \ast 2$ from $x = {}^{-}2$ to $x = 2$, and draw the secant slope lines for the case $s = .5$ at $x = {}^{-}2, {}^{-}1.5, {}^{-}1, {}^{-}.5, 0, .5, 1.0$, and 1.5.

(b) Determine the slopes of the secant lines and make a table of their values.

(c) Write an expression for the secant slope function $(D_{.5}\,F)\,x$, tabulate its values for the eight points of part (a), and compare with the slopes determined in part (b).

5.4 For each of the functions $F\,x$ listed below:

(a) Determine the secant slope function $D_s F$ for $s = 1, \frac{1}{4}$, and $\frac{1}{16}$.

(b) Tabulate the secant slope functions of part (a) for $x = {}^{-}1, 0, 1$, and 2.

(c) Determine the slope function $(D\,F)\,x$.

List of functions:

(i) $F\,x \equiv x \ast 3$
(ii) $F\,x \equiv 2 \times x \ast 3$
(iii) $F\,x \equiv 2 + x \ast 3$
(iv) $F\,x \equiv (x \ast 2) + x \ast 3$
(v) $F\,x \equiv \left(0, \frac{1}{2}, \frac{1}{2}\right) \Pi\, x$
(vi) $F\,x \equiv \left(0, \frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right) \Pi\, x$

(vii) $F\,x \equiv x$
(viii) $F\,x \equiv 3 \times x$
(ix) $F\,x \equiv 3$

5.5 For each of the functions $x \ast 2$, $x \ast 3$, and $x \ast 4$:
(a) Determine the slope functions.
(b) Graph the original functions from $x = {}^{-}1.5$ to $x = 1.5$ and use the results of part (a) to draw tangents for $x = {}^{-}1.5, {}^{-}1.0, {}^{-}0.5, 0, 0.5, 1.0$, and 1.5.

5.6 (a) Use Equation 5.4 to determine the slope of the function $F\,x \equiv x \ast 5$.
(b) Use Equation 5.3 to determine the slope of $F\,x \equiv x \ast 5$, and compare with the result of part (a).

5.7 If $F\,x \equiv x \ast 2$ and $G\,x \equiv x \ast 3$, show that

(a) $F\,F \equiv (F\,) \times F$
(b) $F\,G \equiv G\,F \equiv (F\,) \times (F\,) \times (F\,) \equiv (G) \times G$
(c) $F\,1 + \equiv 1 + (2 \times\ ) + F$
(d) $G\,1 + \equiv 1 + (3 \times\ ) + (3 \times F) + G$
(e) $(G) \times F \equiv (F) \times G$
(f) $G \times F \equiv G\,G$
(g) $(G) \div F \equiv (\ \ )$
(h) $G \div F \equiv 1 \div G$
(i) $F \div G \equiv 1 \div (F\,) \ast 2$

5.8 (a) If $F\,x \equiv +/\iota\,x$ and $G\,x \equiv +/\,(\iota\,x) \ast 3$, show that $G \equiv (F\,) \times F$ (see Exercise 4.24).
(b) If $F\,x \equiv x \ast 2$, show that $+/\,F\,\iota\,x \equiv \left(0, \frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right) \Pi\, x$.

5.9 Apply the rule for (1) the slope of the sum of two functions, and (2) the slope of the function $x \ast n$ to each of the functions (i), (ii), (iii), and (iv) of Exercise 5.4, and compare the results with those of Exercise 5.4 (c). Note that $2 \times x \ast 3 \equiv (x \ast 3) + x \ast 3$.

5.10 Apply the rule for the slope of the product of two functions as well as the rules used in the preceding exercise to each of the functions (v), (vi), and (viii) of Exercise 5.4 and compare with the results of Exercise 5.4 (c).

5.11 For each of the functions listed below, two or more equivalent forms are listed. For each form use the appropriate rules to derive the slope function $D\,F$, and compare the results.

(a) $F\,x \equiv x \ast 3 \equiv x \times x \ast 2 \equiv x \times x \times x$
(b) $F\,x \equiv x \ast n \equiv x \times x \ast n - 1$
(c) $F\,x \equiv x \ast 5 \equiv (x \ast 2) \times x \ast 3$

(d) $F\,x \equiv 3 \times x \ast 4 \equiv (x \ast 4) + (x \ast 4) + x \ast 4$
(e) $F\,x \equiv 1 + (x \ast 2) - x \times x \equiv 1$

5.12 For each of the polynomials listed below, determine the coefficient vector $\boldsymbol{p}$ such that $\boldsymbol{p}\,\Pi\,x \equiv (D\,F)\,x$. Where two equivalent forms are given, compute both and compare the results.

(a) $F\,x \equiv \left(0\,, \frac{1}{2}\,, \frac{1}{2}\right) \Pi\, x$
(b) $F\,x \equiv \left(0\,, \frac{1}{6}\,, \frac{1}{2}\,, \frac{1}{3}\right) \Pi\, x$
(c) $F\,x \equiv ((3 \times x^4) - 2 \times x^2) + (7 \times x) - 3$
(d) $F\,x \equiv (1\,,0\,,0\,,0\,,{}^-1)\,\Pi\,x \equiv (1 - x) \times (0 \neq \iota\,4)\,\Pi\,x$
(e) $F\,x \equiv (2\,,5\,,{}^-8\,,10\,,7\,,{}^-10\,,12)\,\Pi\,x$
$\equiv ((1\,,3\,,{}^-2\,,4)\,\Pi\,x) \times (2\,,{}^-1\,,{}^-1\,,3)\,\Pi\,x$
(f) $F\,x \equiv (1\,,2\,,6\,,7)\,\Pi\,x$
$\equiv ((2\,,6\,,4\,,7)\,\Pi\,x) + ({}^-1,{}^-4,2)\,\Pi\,x$
(g) $F\,x \equiv (1 \div !\,0\,,\iota\,4)\,\Pi\,x \equiv \left(1\,,1\,,\frac{1}{2}\,,\frac{1}{6}\,,\frac{1}{24}\right) \Pi\,x$
(h) $F\,x \equiv (1 \div !\,0\,,\iota\,9)\,\Pi\,x$

5.13 (a) Write a program to determine $\boldsymbol{p}$ as a function of $\boldsymbol{c}$ such that

$$\boldsymbol{p}\,\Pi\,x \equiv (D\,\boldsymbol{c}\,\Pi)\,x$$

(b) Write a program to determine $\boldsymbol{p}$ as a function of $\boldsymbol{c}$ such that $\boldsymbol{p}\,\Pi\,x$ is the nth derivative of $\boldsymbol{c}\,\Pi\,x$. Ensure that $\boldsymbol{p} = \boldsymbol{c}$ for the case $n = 0$. ($D\,D\,F$ is the *second derivative* of F, $D\,D\,D\,F$ is the *third derivative*, and so forth.)

5.14 Consider the problem of Figure 5.10 for a rectangular sheet 3 feet long and 2 feet wide, and determine the value of x that yields the maximum volume.

5.15 The function $P\,t$ of Figure 5.11 (for the position of a freely falling body as a function of time) is

$$P\,t \equiv 16 \times t \ast 2$$

(a) Determine the function $D_s\,P$ for the cases $s = 1, \frac{1}{2}$, and $\frac{1}{4}$.
(b) Determine the slope function $V \equiv D\,P$ (that is, the velocity as a function of time).
(c) Determine the function $D\,D\,P$ (that is, $D\,V$).

5.16 (a) Evaluate the function $\ast\,x$ to three decimal places for values of x from ${}^-2$ to 2 in steps of one-half. (To save work, compute each term by multiplying the preceding term by $\frac{x}{n}$

and by noting that the terms in $\ast\, k$ and $\ast\, -k$ are identical except for sign.)

(b) Using the results of part (a), compute the products $(\ast\, k) \times \ast\, -k$ for values of k from 0.5 to 2 in steps of one-half.

(c) Corroborate the general result suggested by part (b) by computing the first few coefficients of the polynomial for $(\ast\, x) \times \ast\, -x$.

5.17 Since the slope of $\ast\, x$ is equal to $\ast\, x$ for all values of x, the value of the secant slope $(D_s \ast)\, x \equiv \dfrac{(\ast\, x+s) - \ast\, x}{s}$ should be approximately equal to $\ast\, x$ for small values of s.

(i) Compute $(D_s \ast)\, x$ for $x = 1$ and for $s = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$, and compare with the value of $\ast\, 1$.

(ii) Compute $(D_{.1} \ast)\, x$ for $x = 1$, 2, and 3, and compare with $\ast\, 1$, $\ast\, 2$, and $\ast\, 3$, respectively.

5.18 Let $\boldsymbol{x} \equiv s \times \iota\, 5$, let $\boldsymbol{t} \equiv (1 < \iota\, 5)\, /\, \ast\, \boldsymbol{x}$, and let $\boldsymbol{h} \equiv (5 > \iota\, 5)\, /\, \ast\, \boldsymbol{x}$.

(i) For $s = 0.1$, compute $\boldsymbol{t}$ and $\boldsymbol{h}$ correct to four decimal places.

(ii) Compute the difference $\boldsymbol{d} \equiv \boldsymbol{h} - (\boldsymbol{t} - \boldsymbol{h}) \div s$, and explain why the components of $\boldsymbol{d}$ are so small (see Exercise 5.17).

(iii) Write a program to determine the vector $\boldsymbol{d}$ as a function of s, using k terms of the polynomial for $\ast$.

5.19 The results of Exercise 5.16 suggest the following identity:

$$(\ast\, x) \times \ast\, -x \equiv 1$$

In other words, $(\ast\, -x)$ is the reciprocal of $\ast\, x$. Prove this identity by the following steps:

(a) Show that the function $H\, x \equiv (\ast\, x) \times \ast\, -x$ is a constant. (HINT: Show that the slope of $H\, x$ is zero, using the product rule and Equation 5.10 with $r = {}^{-}1$.)

(b) Show that the constant function $H\, x$ has the value 1 by evaluating it for some value of x.

5.20 Evaluate the functions $A\, x$ and $B\, x$ for $x = {}^{-}2$ to $x = 2$ in steps of 0.5 (use the terms computed in Exercise 5.16).

5.21 (a) Repeat Exercise 5.17, substituting $(D_s A)\, x$ for $(D_s \ast)\, x$ and comparing it with $B\, x$ rather than with $\ast\, x$.

(b) Repeat Exercise 5.17, substituting $(D_s B)\, x$ for $(D_s \ast)\, x$ and comparing it with $A\, x$.

5.22 (a) Repeat Exercise 5.18, substituting $A\,\boldsymbol{x}$ for $*\,\boldsymbol{x}$ and redefining $\boldsymbol{d}$ as $\boldsymbol{d} \equiv ((5 > \iota\,5)\,/\,B\,\boldsymbol{x}) - (\boldsymbol{t} - \boldsymbol{h}) \div s$.

(b) Repeat part (a), interchanging the roles of the functions A and B.

5.23 Write programs to evaluate each of the following functions to an accuracy such that the absolute value of the first neglected term is less than a specified tolerance t.

(a) $*\,x$

(b) $A\,x$

(c) $B\,x$

5.24 Write a single program having arguments $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ which evaluates $*\,\boldsymbol{x}_1$ if $\boldsymbol{x}_2 \equiv 0$; $A\,\boldsymbol{x}_1$ if $\boldsymbol{x}_2 \equiv 1$; and $B\,\boldsymbol{x}_1$ if $\boldsymbol{x}_2 \equiv {}^{-}1$, all evaluations being performed so that the absolute value of the first neglected term is less than a specified tolerance $\boldsymbol{x}_3$.

5.25 Write a program to check the theorem suggested by the results of Exercise 5.16 (b) for further values of k.

5.26 (a) Let $e = *\,1 = 2.718$ as determined in Exercise 5.16. Compute the following powers of e: e^{-2}, $e^{-\frac{3}{2}}$, e^{-1}, $e^{-\frac{1}{2}}$, e, $e^{\frac{1}{2}}$, e^{1}, $e^{\frac{3}{2}}$, e^{2}, and compare them with the values of $*\,{}^{-}2$, $*\,-\frac{3}{2}$, $*\,{}^{-}1$, and so forth, obtained in Exercise 5.16.

(b) Write a program to check the theorem suggested by the results of part (a) for further values.

5.27 (a) Compute and compare the values of $*\,x + y$ and $(*\,x) \times *\,y$ for a few values of x and y.

(b) Write a program to further test the result suggested by part (a).

5.28 Establish the identity suggested in Exercise 5.27 by considering the expression:

$$*\,x + y \equiv 1 + (x + y) + \frac{(x+y)^2}{!\,2} + \frac{(x+y)^3}{!\,3} + \frac{(x+y)^4}{!\,4} + \ldots$$

and collecting all terms in x^0, all terms in x^1, and so forth, to show that

$$*\,x + y \equiv \left(1 + y + \frac{y^2}{!\,2} + \frac{y^3}{!\,3} + \ldots\right) + x \times \left(1 + y + \frac{y^2}{!\,2} + \frac{y^3}{!\,3} + \ldots\right) +$$
$$\frac{x^2}{!\,2} \times \left(1 + y + \frac{y^2}{!\,2} + \frac{y^3}{!\,3} + \ldots\right) +$$

$$\frac{x^3}{!3}\times\left(1+y+\frac{y^2}{!2}+\frac{y^3}{!3}+\ldots\right)+\ldots$$

$$\equiv\left(1+y+\frac{y^2}{!2}+\frac{y^3}{!3}+\ldots\right)\times\left(1+x+\frac{x^2}{!2}+\frac{x^3}{!3}+\ldots\right)$$

$$\equiv(*\,y)\times *\,x$$

5.29 (a) Evaluate the first few coefficients of the polynomial

$$((C\,x)\times C\,x)+(S\,x)\times S\,x$$

and compare them with the result of Equation 5.17.

(b) Evaluate the first few coefficients of the polynomial

$$((A\,x)\times A\,x)-(B\,x)\times B\,x$$

and conjecture a relation similar to that of Equation 5.17.

(c) Prove the validity of the relation conjectured in part (b).

5.30 (a) Repeat Exercise 5.17, substituting $(D_s C)\,x$ for $(D_s *)\,x$ and comparing with $-S\,x$ rather than with $*\,x$.

(b) Repeat Exercise 5.17, substituting $(D_s S)\,x$ for $(D_s *)\,x$ and comparing with $C\,x$.

5.31 (a) Repeat Exercise 5.18, substituting $C\,x$ for $*\,x$ and redefining $\boldsymbol{d}$ as $\boldsymbol{d}\equiv((5>\iota\,5)\,/-S\,\boldsymbol{x})-(\boldsymbol{t}-\boldsymbol{h})\div s$.

(b) Repeat Exercise 5.18, substituting $S\,x$ for $*\,x$ and redefining $\boldsymbol{d}$ as $\boldsymbol{d}\equiv((5>\iota\,5)\,/\,C\,\boldsymbol{x})-(\boldsymbol{t}-\boldsymbol{h})\div s$.

5.32 For each of the following polynomials of unlimited degree, make a table showing the sum of the first n terms for values of n from 1 to 6 and showing (if possible) the complete sum as given by Equation 5.18.

(a) $(1,1,1,1,1,1,1,\ldots)\,\Pi\,.5\equiv 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$
(b) $(1,1,1,\ldots)\,\Pi\,.9$
(c) $(1,1,1,\ldots)\,\Pi\,.1$
(d) $(4,4,4,\ldots)\,\Pi\,.5$
(e) $(1,{}^{-}1,1,{}^{-}1,1,{}^{-}1,\ldots)\,\Pi\,.5$
(f) $(1,1,1,1,\ldots)\,\Pi\,1$
(g) $(1,{}^{-}1,1,{}^{-}1,1,{}^{-}1,\ldots)\,\Pi\,1$

5.33 Throughout this exercise use Equation 5.19 to determine error bounds.

(a) Use Equation 5.14 to compute the values (to three digits of accuracy) of the functions $C\,x$ and $S\,x$ for the following values of x: $-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}$.

(b) Write programs to evaluate $S\,x$ and $C\,x$ to a prescribed tolerance t, and use them to compute $S\,\pi$, $C\,\pi$, $S-\pi$, and $C-\pi$.

(c) Write a single program (in the manner of Exercise 5.24) to evaluate any one of the functions $\ast$, A, B, C, and S.

5.34 (a) Without making any further evaluations of the functions $C\,x$ and $S\,x$, make as accurate a graph of them as possible using the values computed in Exercise 5.33, Equation 5.17 (to establish upper and lower bounds on the values of the functions), Equation 5.15 (to draw tangents to the curves at the points computed in Exercise 5.33), and the fact that C is an even function and that S is odd.

(b) Use the programs of Exercise 5.33 to evaluate the functions C and S at about forty equally spaced points from $^{-}4$ to 4, graph the results, and compare them with the graphs produced in part (a).

5.35 Determine the areas enclosed by the x-axis, the curve $F\,x$, and the lines $x=a$ and $x=t$, for

(a) $F\,x \equiv x \ast 2$

(b) $F\,x \equiv 4 \times x \ast 3$

(c) $F\,x \equiv (1\,,1\,,1)\;\Pi\,x$

(d) $F\,x \equiv \mathbf{c}\,\Pi\,x$

5.36 Show that the function $G\,t$ of Figure 5.15 represents the area between the curve F and the x-axis even for values of $t > 1.5$ (where F is negative), provided that the area enclosed by the negative portion of the curve F is treated as a *negative* area.

5.37 Determine the entire area enclosed by the x-axis and the positive portion of the curve of the function $F\,x \equiv 4 - x^2$.

5.38 (a) Determine the volume of revolution of the curve $F\,x \equiv x \ast 3$ enclosed between planes at $x=1$ and $x=2$.

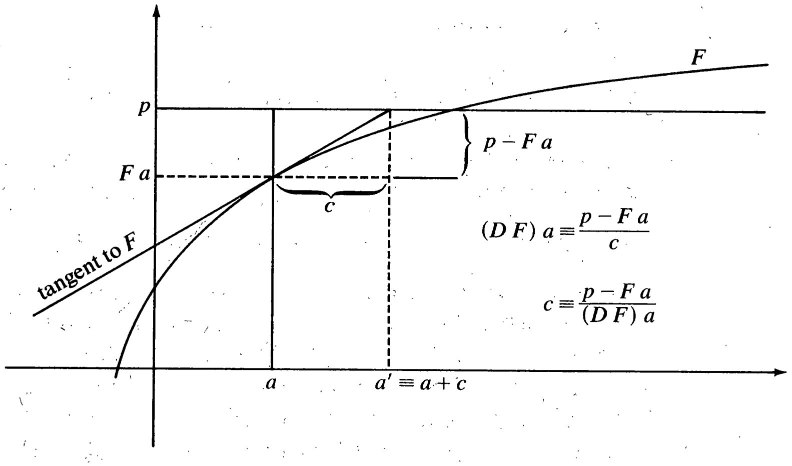
(b) Determine the volume of a hemisphere of radius r. (Assume that the center is placed at the origin.)

5.39 Exercise 2.14 (d) shows how the equation $a \ast n \equiv p$ can be solved for a by computing successively better approximations to a. The same method can be applied to solve

$$F\,a \equiv p$$

for any other function F.

A much faster method (requiring fewer approximations) can be derived by using the slope of F in the calculation of a new approximation. From the accompanying illustration it is clear that



$$c \equiv \frac{p - F\,a}{(D\,F)\,a}$$

is a good correction. Hence

$$a \leftarrow a + (p - F\,a) \div (D\,F)\,a$$

yields an improved approximation.

(a) Revise Program 2.6 to apply this method.

(b) Write a program using this method to determine a real zero of the polynomial $\boldsymbol{c} \,\Pi\, x$ (that is, a real number x such that $\boldsymbol{c} \,\Pi\, x \equiv 0$). Assume that an initial approximation $\boldsymbol{b}_1$ and a tolerance $\boldsymbol{b}_2$ are given.

(c) Write a program to determine the value of x such that $S\,x \equiv 1$ to within a tolerance t. Execute the program for $t \equiv .00001$ using $x \equiv 3.14 \div 2$ as an initial approximation.

5.40 The abbreviated notation introduced for composite functions was limited to functions of a single argument. Hence $F \equiv +\ 1\ \div$ indicates that $F\,x \equiv x + 1 \div x$. The notion can be extended to functions of many variables by simply requiring that the successive positions to be filled by arguments be filled by *distinct* arguments. Then $F \equiv +\ 1\ \div$ would indicate that F is a function of two arguments and that $x\,F\,y \equiv x + 1 \div y$. Using this definition of F, compute the following:

(a) $F\,/\,\boldsymbol{x}$ for $\boldsymbol{x} = 1, 2, 3$

(b) $F / \iota n$, for values of n from 1 to 6
(c) $F / 1, 1, 1, 1$
(d) $F / 0 \neq \iota n$, for values of n from 1 to 8
(e) $F / 3, 7, 15, 1, 256$

5.41 Using the scheme of Exercise 5.40 to define dyadic functions, the expression $+ 1 \div$ represents a dyadic function and therefore $(+ 1 \div) / 2, 3, 4 \equiv 2 + 1 \div 3 + 1 \div 4 \equiv \frac{30}{13}$. If $\boldsymbol{c}$ is any vector, then $(+ 1 \div) / \boldsymbol{c}$ is called the *continued fraction* defined by the vector $\boldsymbol{c}$ (see, for example, C. D. Olds, *Continued Fractions*, L. W. Singer, 1963).

(a) Show that if the components of $\boldsymbol{c}$ are all positive, then each of the successive values of $(+ 1 \div) / (j \geq \iota \rho \boldsymbol{c}) / \boldsymbol{c}$ for $j = 1, 2, 3, \ldots$ fall between the preceding pair.
(b) Show that $(+ 1 \div) / \iota n$ approaches a limiting value as n increases.

5.42 If $\boldsymbol{f}$ is a vector such that $\boldsymbol{f}_1 \equiv \boldsymbol{f}_2 \equiv 1$ and $\boldsymbol{f}_{j+1} \equiv \boldsymbol{f}_j + \boldsymbol{f}_{j-1}$ for $j > 1$, then $\boldsymbol{f}$ is called a vector of *Fibonacci numbers.* Let $\boldsymbol{h} \equiv (\rho \boldsymbol{f}) > \iota \rho \boldsymbol{f}$ and let $\boldsymbol{t} \equiv 1 < \iota \rho \boldsymbol{f}$.

(a) Compute the value of $\boldsymbol{f}$ for $\rho \boldsymbol{f} \equiv 9$.
(b) Prove that $1, \boldsymbol{f} + 0, \boldsymbol{h} / \boldsymbol{f}$ is also a Fibonacci vector.
(c) Compute the vector $(\boldsymbol{t} / \boldsymbol{f}) \div \boldsymbol{h} / \boldsymbol{f}$ and compare its components with the results of Exercise 5.40 (d).
(d) Prove the relation suggested by part (c).

Chapter Six

Circular Functions

The Sine and Cosine Functions

If P is a point on a circle with center at the origin and with a radius of 1 unit, then the length of arc measured counterclockwise from the point (1, 0) to the point P is called the *arc* of the point P. In Figure 6.1, for example, the arc of P is 1 and the arc of Q is 0.5.

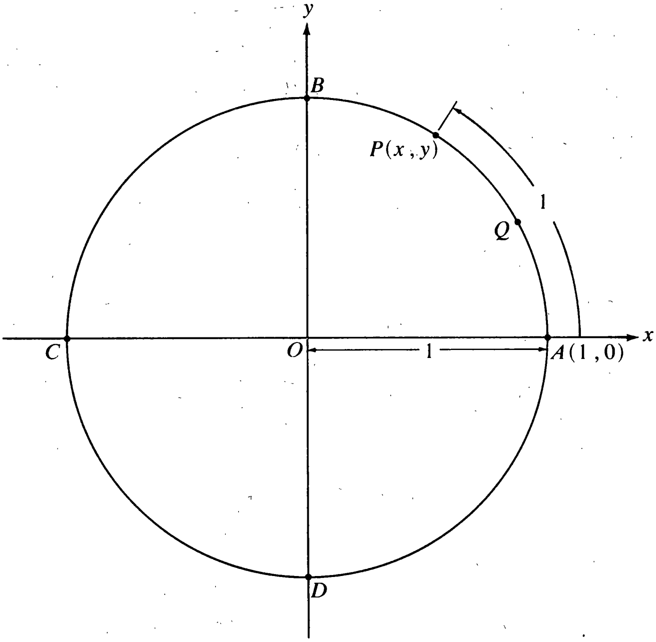


Figure 6.1 The arc of a point on the unit circle

If P is the point (x, y) and a is the arc of P, then the values of the coordinates x and y are uniquely determined by the value of a. In other words, x and y are functions of a. The function which determines the vertical component of P is called the *sine* function, and the function which determines the horizontal component is called the *cosine*. Thus $y \equiv$ sine a and $x \equiv$ cosine a. The terms *sine* and *cosine* are often abbreviated as *sin* and *cos* respectively.

The length of a semicircle of unit radius is denoted by the symbol π and is approximately equal to 3.14159. Consequently, the arcs of the points A, B, C, and D of Figure 6.1 are $0, \frac{\pi}{2}, \pi$, and $\frac{3}{2} \times \pi$, respectively. The following relations are therefore evident:

$$\begin{array}{ll} \cos 0 \equiv 1 & \sin 0 \equiv 0 \\ \cos \frac{\pi}{2} \equiv 0 & \sin \frac{\pi}{2} \equiv 1 \\ \cos \pi \equiv {}^{-}1 & \sin \pi \equiv 0 \\ \cos \frac{3}{2} \times \pi \equiv 0 & \sin \frac{3}{2} \times \pi \equiv {}^{-}1 \end{array}$$

An arc measured *clockwise* from the reference point $(1, 0)$ is considered to be negative. Hence the arc of point D can be considered as $-\frac{\pi}{2}$ as well as $\frac{3}{2} \times \pi$. Likewise the arcs of C and B are $-\pi$ and $-\frac{3}{2} \times \pi$. Consideration of the points E and F and the points G and H of Figure 6.2 shows that in general (because of symmetry about the x-axis)

$$\left.\begin{array}{l} \cos - a \equiv \cos a \\ \sin - a \equiv - \sin a \end{array}\right\} \qquad (6.1)$$

In other words, the cosine function is even and the sine function is odd.

As illustrated in Figure 6.3, an arc of length greater than $2 \times \pi$ (that is, one circumference) can be measured off. From the figure it is clear that $\cos\ (a + 2 \times \pi) \equiv \cos a$ and that $\sin\ (a + 2 \times \pi) \equiv \sin a$. More generally, if n is *any* integer (positive, negative, or zero), then

$$\left.\begin{array}{l} \cos a + n \times (2 \times \pi) \equiv \cos a \\ \sin a + n \times (2 \times \pi) \equiv \sin a \end{array}\right\} \qquad (6.2)$$

The sine and cosine functions therefore repeat themselves at intervals of $2 \times \pi$ and are said to be *periodic* functions with *period* $2 \times \pi$.

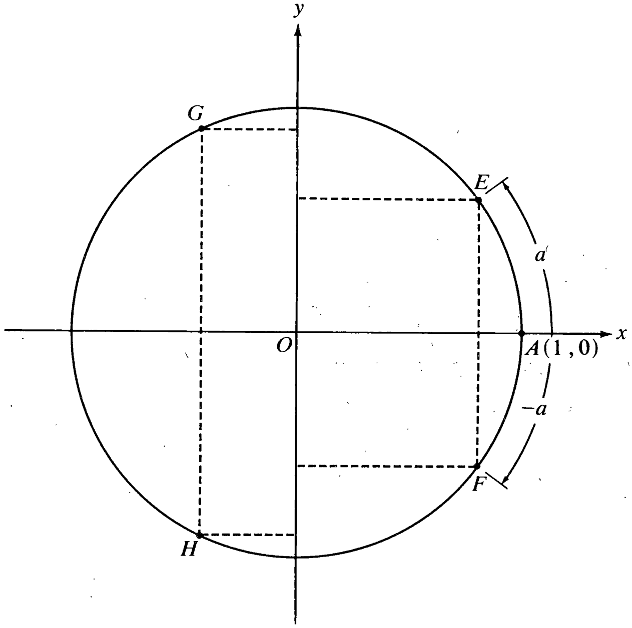


Figure 6.2 The even character of the cosine and the odd character of the sine

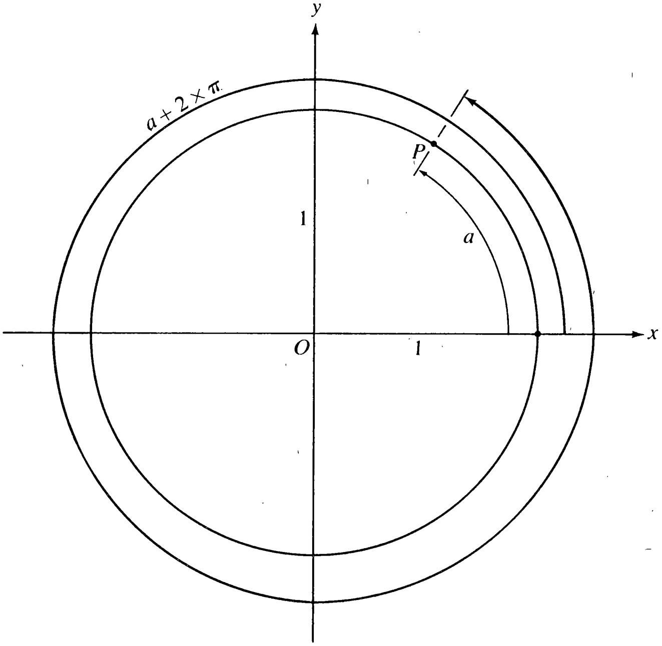


Figure 6.3 An arc exceeding $2 \times \pi$

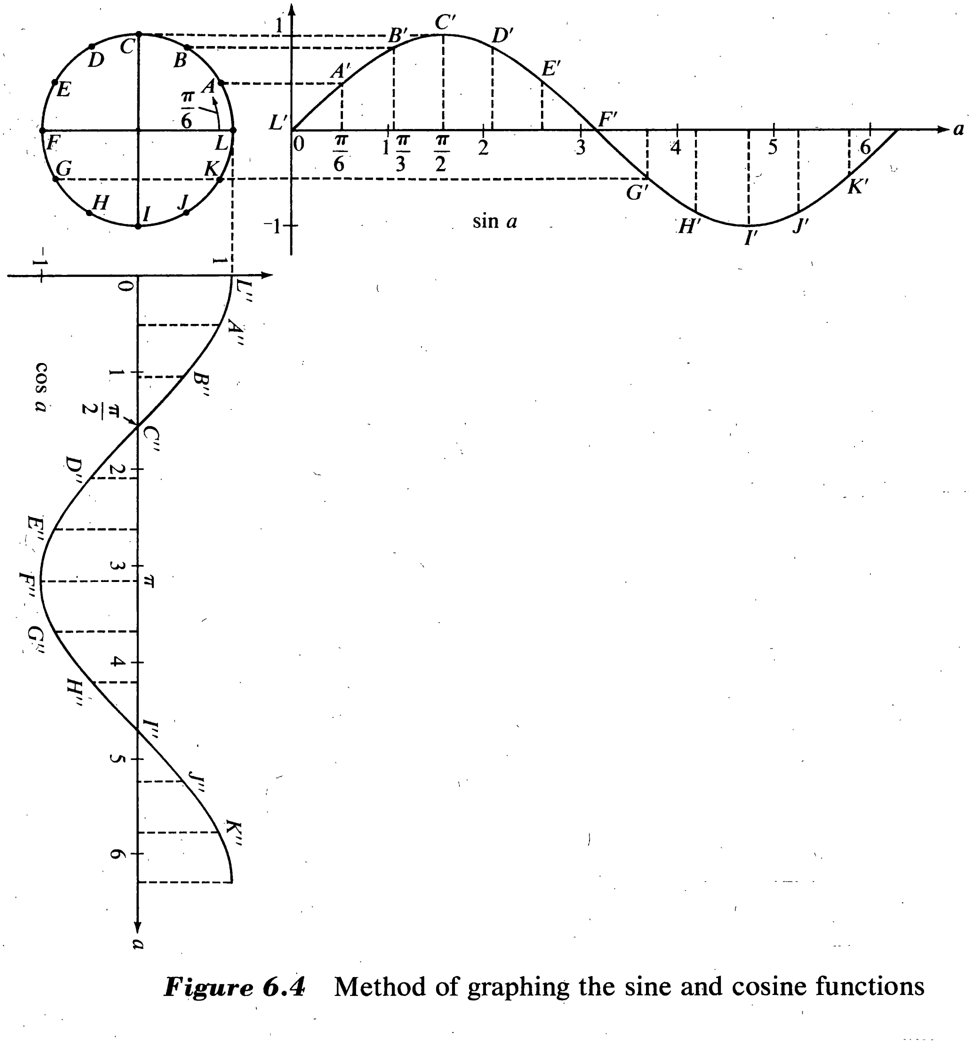


Figure 6.4 Method of graphing the sine and cosine functions

The sine and cosine functions can be graphed by the method of Figure 6.4. Figure 6.5 shows the graphs superimposed and extended to negative values of the argument. The even character of the cosine, the odd character of the sine (Equation 6.1), and the periodicity of both (Equation 6.2) are evident from Figure 6.5.

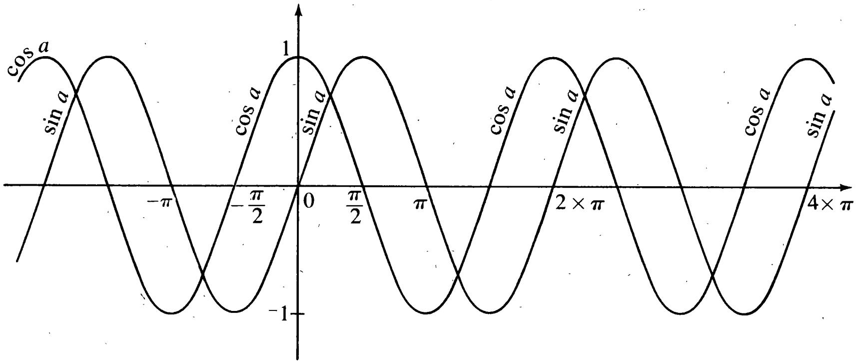


Figure 6.5 The sine and cosine functions

Figure 6.5 also shows that the graphs of sine and cosine are displaced horizontally by $\frac{\pi}{2}$. More precisely, $\sin a \equiv \cos a - \frac{\pi}{2}$. Since cosine is an even function, this can also be written as

$$\sin a \equiv \cos \frac{\pi}{2} - a \tag{6.3}$$

Arcs a and b are said to be *complementary* if their sum is $\frac{\pi}{2}$. The arcs a and $\frac{\pi}{2} - a$ are clearly complementary; Equation 6.3 therefore shows why the term *co*sine is used for the second of the circular functions. Moreover $\sin \frac{\pi}{4} \equiv \cos \frac{\pi}{2} - \frac{\pi}{4} \equiv \cos \frac{\pi}{4}$, as can also be seen in Figure 6.6 from the symmetry about the line through P and the origin.

The application of the Pythagorean theorem to Figure 6.7 reveals another important property of the circular functions:

$$(\sin a)^2 + (\cos a)^2 \equiv 1 \tag{6.4}$$

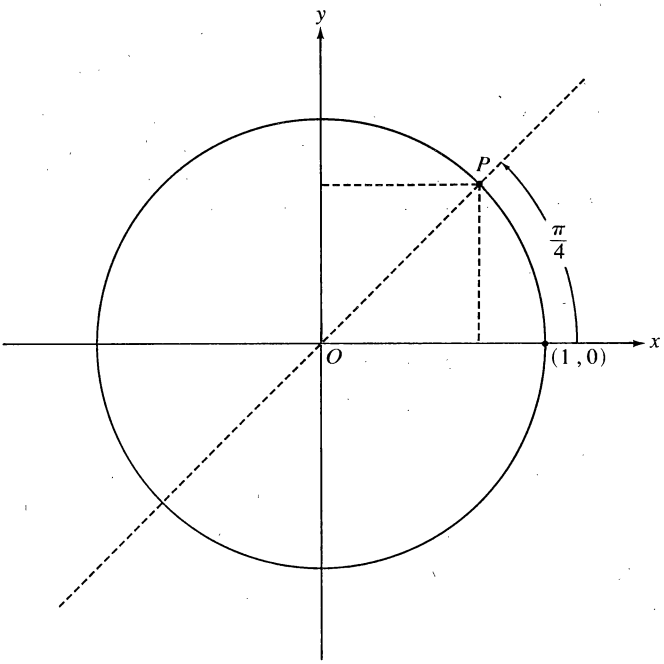


Figure 6.6 Equality of the sine and cosine of $\frac{\pi}{4}$

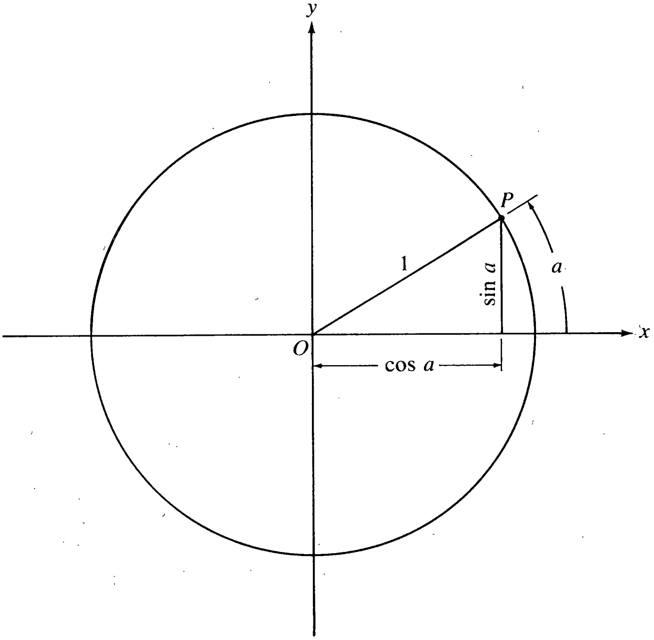


Figure 6.7 $(\sin a)^2 + (\cos a)^2 \equiv 1$

Since $\sin\frac{\pi}{4} \equiv \cos\frac{\pi}{4}$, Equation 6.4 shows that $\sin\frac{\pi}{4} \equiv \cos\frac{\pi}{4} \equiv \sqrt{0.5}$.

Figure 6.7 also shows that the sine and cosine functions can be used to describe the relations between the sides of a right triangle having a hypotenuse of length 1. They can also be used to describe the relations in a right triangle of arbitrary size, as shown in Figure 6.8. Since the triangles *AOC* and *BOD* are similar, the corresponding sides are proportional:

$$\frac{OB}{OA} = \frac{BD}{AC} = \frac{OD}{OC}$$

Since $OA = 1$, $AC = \sin a$, and $OC = \cos a$, then

$$BD = OB \times \sin a$$

and

$$OD = OB \times \cos a$$

It is frequently convenient to treat a triangle such as *BOD* without explicit reference to its intersections with the unit circle having center *O*. For this reason one speaks of the *angle formed by the lines DO and BO* and denotes this angle by $\angle DOB$. However, the *measure* of this angle is the length of the arc it subtends on the unit circle having center *O*. The measure of an angle is frequently indicated by writing its value beside a curved arrow between the sides of the angle, as in Figure 6.8. This curved arrow stands for the arc of the unit circle subtended by the angle.

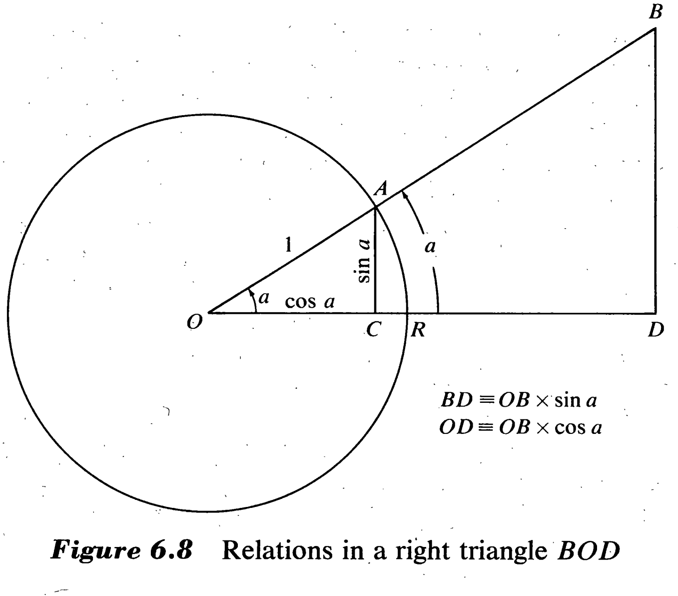


Figure 6.8 Relations in a right triangle *BOD*

Since an argument of either of the circular functions is the measure of some angle, this argument itself is frequently referred to as an *angle*. It must be emphasized, however, that the terminology and notation for angles is merely an (often confusing) abbreviation: "$\angle DOB$" or "the measure of $\angle DOB$" simply means the length of arc subtended by the lines DO and BO intersecting the unit circle having center O.

Because the arc which determines the measure of an angle must be measured in the same metric units in which the radius has length 1, the unit of measure of an angle is called a *radian*. Thus the measure of a right angle is $\frac{\pi}{2}$ *radians*. Measures of angles are often expressed in *degrees*, with 360 degrees corresponding to $2 \times \pi$ radians. A measurement expressed in degrees is usually indicated by a small raised circle or *degree sign*; thus $\frac{\pi}{4}$ radians corresponds to 45°. Radian measure will be used throughout the present treatment except when dealing with standard tables of the circular functions, which are commonly expressed in degrees.

(Do Exercises 6.1 – 6.5.)

Addition Theorems for the Sine and Cosine

An identity for the value of $F\,x+s$ is called an *addition theorem* for the function F. An addition theorem is very useful in the study of a function. In particular, it is useful in evaluating the expression $(F\,x+s)-F\,x$ which occurs in the expression for the secant slope of F. The binomial theorem used in deriving the slope of the function $x * n$ is an example of an addition theorem.

Addition theorems for the sine and cosine will now be derived with the aid of Figure 6.9. The line segments OR, OM, and ON are radii of a circle of radius 1, and $\angle ROM$, $\angle MON$, and $\angle RON$ have measures of x, y, and $x+y$ radians respectively. The segments NV and QT are dropped perpendicular to OR, NQ is perpendicular to OM, and QP is perpendicular to NV.

The angles PNQ and PQO are equal, since each is complementary to NQP; and angles PQO and ROM are equal, since they are alternate interior angles with respect to two parallel lines and a transversal. Hence $\angle PNQ = \angle ROM$, and $\angle PNQ$ therefore has the measure x, as indicated in the figure.

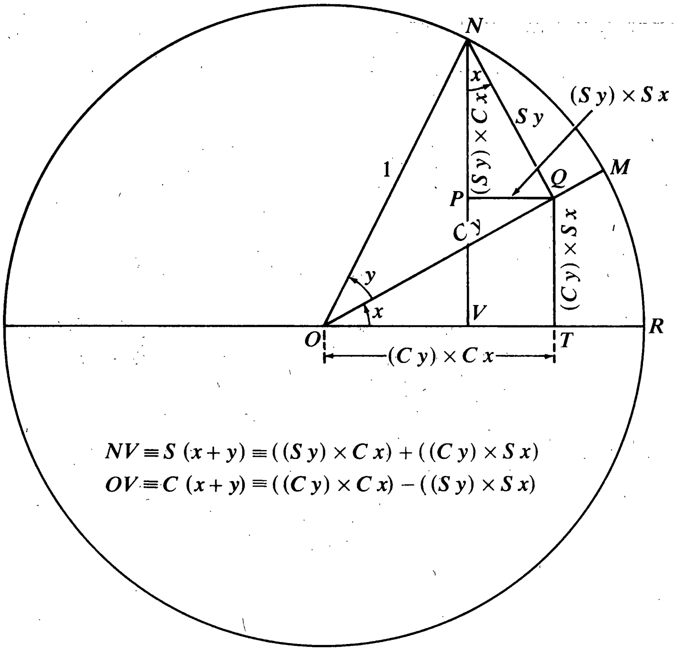


Figure 6.9 Addition theorems for the sine and cosine functions

Using *MO* as the reference line, it is clear that the length of segment *NQ* is equal to sin *y* (abbreviated *S y* in the figure). Likewise the length of segment *OQ* is equal to cos *y* (abbreviated *C y* in the figure).

Using line *PN* as a reference line and using the results displayed in Figure 6.8, it is clear that the length of segment *NP* is $(\sin y) \times \cos x$ and the length of segment *PQ* is $(\sin y) \times \sin x$. A similar argument shows that the segments *QT* and *OT* have the lengths indicated in the figure.

Finally, the length of *NV* is $\sin\,(x + y)$, and since it is also equal to the sum of the lengths of *NP* and *QT*, it follows that

$$\sin x + y \equiv ((\sin y) \times \cos x) + (\cos y) \times \sin x \qquad (6.5)$$

Similarly

$$\cos x + y \equiv OV \equiv OT - QP$$

and hence

$$\cos x + y \equiv ((\cos y) \times \cos x) - (\sin y) \times \sin x \qquad (6.6)$$

(Do Exercises 6.6–6.10.)

The Slope Functions of the Sine and Cosine

The slope of the sine function can now be obtained by using the methods of Chapter 5 and the addition theorem (Equation 6.5):

$$
\begin{aligned}
D_s \sin x &\equiv \frac{(\sin x + s) - \sin x}{s} \\
&\equiv \frac{(((\sin s) \times \cos x) + (\cos s) \times \sin x) - \sin x}{s} \\
&\equiv \left(\frac{\sin s}{s} \times \cos x\right) - \frac{1 - \cos s}{s} \times \sin x \qquad (6.7)
\end{aligned}
$$

The slope function $(D \sin)\, x$ can be obtained from the foregoing expression for the secant slope by determining the limiting values of the factors $\frac{\sin s}{s}$ and $\frac{1 - \cos s}{s}$ as s approaches zero.

From Figure 6.10 it is clear that the lengths of the arc RA, the line segment NA, and the tangent TA satisfy the following relations:

$$NA \leq RA \leq TA$$

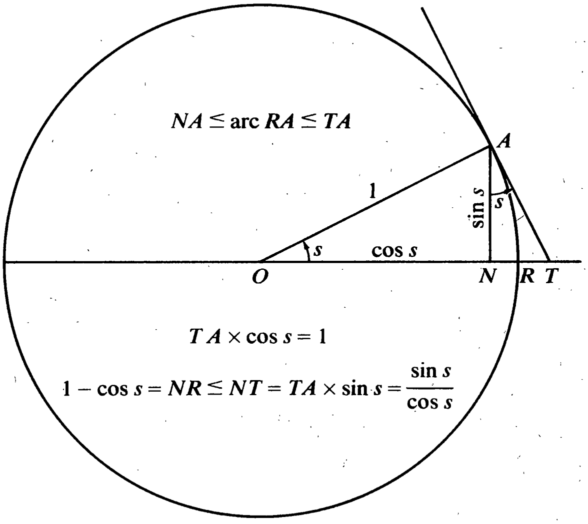


Figure 6.10 The limiting values of $\frac{\sin s}{s}$ and $\frac{1 - \cos s}{s}$ as s approaches 0

Therefore

$$\frac{\sin s}{NA} \geq \frac{\sin s}{RA} \geq \frac{\sin s}{TA}$$

But $NA = \sin s$, and $RA = s$, and $TA = \dfrac{\sin s}{\cos s}$ (since $TA = OT \times \sin s$ and $OT \times \cos s = 1$). Hence

$$1 \geq \frac{\sin s}{s} \geq \cos s$$

The ratio $\dfrac{\sin s}{s}$ therefore lies between 1 and $\cos s$. But as s approaches 0, $\cos s$ approaches 1, and hence the limiting value of $\dfrac{\sin s}{s}$ must be 1.

Similarly, $\dfrac{1 - \cos s}{s} = \dfrac{NR}{s} \leq \dfrac{NT}{s}$ and (since $\angle NAT = s$)

$$\frac{NT}{s} = \frac{TA \times \sin s}{s} = \frac{\sin s}{\cos s} \times \frac{\sin s}{s}$$

Therefore $\dfrac{1 - \cos s}{s} \leq \dfrac{\sin s}{\cos s} \times \dfrac{\sin s}{s}$. But as s approaches 0, $\dfrac{\sin s}{s}$ and $\cos s$ each approach 1, and $\sin s$ approaches 0. Hence the limiting value of $\dfrac{1 - \cos s}{s}$ is 0.

To summarize:

$$\left.\begin{array}{l} \dfrac{\sin s}{s} \text{ approaches 1 as } s \text{ approaches 0} \\ \dfrac{1 - \cos s}{s} \text{ approaches 0 as } s \text{ approaches 0} \end{array}\right\} \tag{6.8}$$

These results can now be substituted in Equation 6.7 to yield the slope function of $\sin x$:

$$(D \sin)\, x \equiv \cos x \tag{6.9}$$

This simple result can be corroborated by the graphs of Figure 6.5. The slope of $\sin a$ for any value of a is seen to be equal to the value of the cosine for that same value of a. In particular, for $a = \dfrac{\pi}{2}$ the slope of $\sin a$ is 0, as is the value of $\cos a$; and for $a = \pi$, the slope of $\sin a$ is $^{-}1$, as is the value of $\cos a$.

The slope function of $\cos x$ can now be obtained by an analogous use of Equation 6.6:

$$
\begin{aligned}
(D_s \cos)\, x &\equiv \frac{(\cos x + s) - \cos x}{s} \\
&\equiv \frac{(((\cos s) \times \cos x) - (\sin s) \times \sin x) - \cos x}{s} \\
&\equiv \left(\left(\frac{(\cos s) - 1}{s}\right) \times \cos x\right) - \left(\frac{\sin s}{s}\right) \times \sin x
\end{aligned}
$$

Again using Equation 6.8 for the limiting values of the factors involving s, the slope of $\cos x$ becomes

$$(D \cos)\, x \equiv - \sin x \tag{6.10}$$

Equations 6.10 and 6.9 can be rewritten as follows:

$$\left.\begin{aligned} D \cos &\equiv - \sin \\ D \sin &\equiv \cos \end{aligned}\right\} \tag{6.11}$$

Consequently

(Do Exercise 6.11.)

$$\left.\begin{aligned} D\, D \cos &\equiv D\,(- \sin) \equiv - \cos \\ D\, D \sin &\equiv D \cos \equiv - \sin \end{aligned}\right\} \tag{6.12}$$

Polynomial Approximations for Sine and Cosine

The patterns of Equations 6.11 and 6.12 were encountered in Chapter 5, in the study of the polynomials $C\,x$ and $S\,x$. The relevant equations (5.15 and 5.16) are repeated here for comparison:

$$\left.\begin{aligned} D\,C &\equiv - S \\ D\,S &\equiv C \end{aligned}\right\} \tag{5.15}$$

$$\left.\begin{aligned} D\,D\,C &\equiv - C \\ D\,D\,S &\equiv - S \end{aligned}\right\} \tag{5.16}$$

From Equations 5.16 and 6.12 it is evident that the functions S and sine are similar in behavior, as are C and cosine. Two functions can have the same slope functions and yet differ by a constant as illustrated in Figure 5.9. Note, however, that $\sin x$ and $S\,x$ agree for $x = 0$, that is, $\sin 0 = S\,0$. Moreover, since $\cos 0 = C\,0 = 1$, Equations 5.15 and 6.11 can be used to show that the slopes of $\sin x$ and $S\,x$ also agree at $x = 0$, that is, $(D \sin)\,0 = (D\,S)\,0$. Similar arguments can be used to show similar agreement between $\cos x$ and $C\,x$.

The functions $\sin x$ and $S\,x$ are in fact identical, as are $\cos x$ and $C\,x$; the notation "cos" and "sin" will now be replaced by the equivalent notation "C" and "S."

The functions C and S must of course satisfy the addition theorems of Equations 6.5 and 6.6. Although these relations were easily derived from geometry, they are not at all obvious from the definition of the polynomials C and S. The theorems can be corroborated, however, by applying them for specific values of the arguments x and y, and also by computing the first few terms of the polynomial for $S\,x+y$ and comparing these terms with the coefficients in the expression indicated in Equation 6.5.

It is also interesting to compare Equations 6.4 and 5.17:

$$(\sin a)^2 + (\cos a)^2 \equiv 1$$
$$(S\,a)^2 + (C\,a)^2 \equiv 1$$

The values of $C\,x$ and $S\,x$ can also be computed and compared with $\cos x$ and $\sin x$ for values of x $\left(\text{such as } \frac{\pi}{4}, \frac{\pi}{2}, \text{ and } -\frac{\pi}{4}\right)$ for which the values of $\cos x$ and $\sin x$ are known.

(Do Exercises 6.12–6.14.)

The Tangent Function

The *tangent* function $T\,x$ is defined as the quotient $(S\,x) \div C\,x$; in abbreviated form

$$T \equiv (S) \div C \qquad (6.13)$$

The reason for the name *tangent* is evident from Figure 6.10, where the length of the tangent segment TA is shown to be equal to $\frac{\sin s}{\cos s}$, that is, to $T\,s$.

The tangent function $T\,x$ can be evaluated by evaluating $S\,x$ and $C\,x$ and computing their quotient. Whereas the sine and cosine functions are bounded by the value 1 (that is, their absolute values never exceed the value 1), the tangent function is unbounded and becomes infinite at points where $C\,x$ is zero.

(Do Exercises 6.15–6.17.)

Tables of the Circular Functions

Brief tables of the sine, cosine, and tangent functions are provided in Appendix B for arguments from 0 to $\frac{\pi}{2}$, that is, for angles from 0 to 90 degrees. More extensive tables, using a smaller interval between successive values of the argument and providing greater precision in the functions, are readily available.

If the value of the argument x falls somewhere between two successive argument values a and b in the table, then $F\,x$ can be determined as a weighted sum of $F\ a$ and $F\ b$ as follows:

$$F\ x \equiv (F\ a) + \left(\frac{x-a}{b-a}\right) \times (F\ b) - F\ a$$

This is called *linear interpolation* because, as shown in Figure 6.11, it evaluates $F\ x$ as if the function between the arguments a and b were a straight line passing through the points $(a\ , F\ a)$ and $(b\ , F\ b)$.

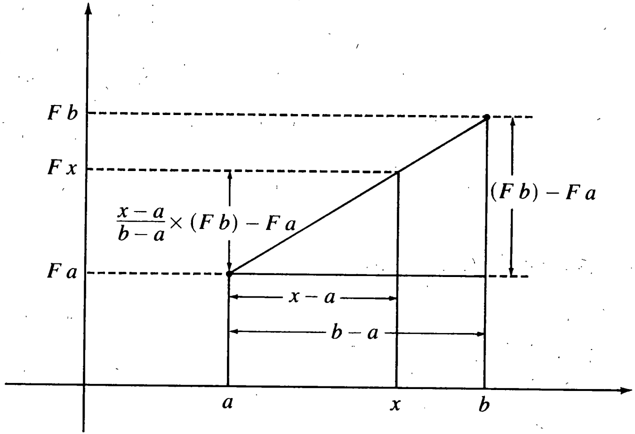


Figure 6.11 Linear interpolation for the function F

For example, the sine of 6.16 degrees is derived from the table entries

$$\text{sine of } 6° = 0.1045$$
$$\text{sine of } 7° = 0.1219$$

as follows:

$$\begin{aligned}\text{sine of } 6.16° &= 0.1045 + \left(\frac{6.16-6.0}{7-6}\right) \times (0.1219-0.1045)\\ &= 0.1073\end{aligned}$$

The table can also be used to determine values of the inverse functions, that is, to determine the angle corresponding to a specified value of sine, cosine, or tangent. For example, if the sine of x is 0.4384, then a search of the column for the sine shows that this value corresponds to an angle of exactly 26°. Interpolation can of course be applied in this inverse use of the tables as well. For example, if the sine of x is .4462, then the nearest table entries above and below are

$$\text{sine of } 26° = .4384$$
$$\text{sine of } 27° = .4540$$

Linear interpolation then gives

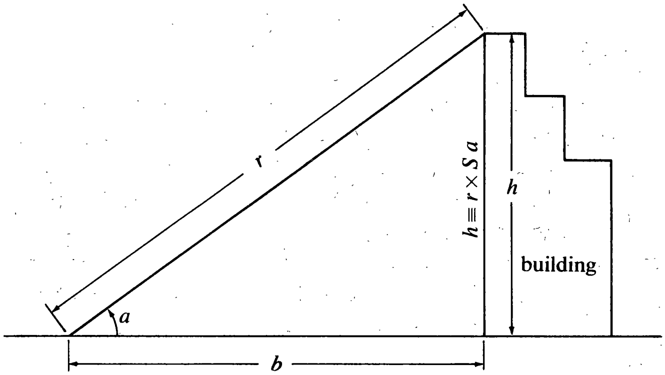
$$x = 26° + \frac{.4462 - .4384}{.4540 - .4384} \times (27° - 26°) = 26.5°$$

(Do Exercises 6.18–6.24.)

Applications of the Circular Functions

Applications of the circular functions fall into two major classes, geometric and nongeometric. The nongeometric applications to be treated are perhaps more interesting but require some knowledge of physics. The treatment of the physical principles employed may be too brief to satisfy some readers. However, since these examples are only intended to illustrate the wide range of application of the circular functions, the reader should follow the examples without worrying about the full significance of the physical concepts introduced.

Geometric applications. Geometric applications concern surveying and related problems. For example, the height h of a building can be calculated from the measured angle a and the horizontal distance b measured from the base, as shown in Figure 6.12. From Figure 6.12 and the relations shown in Figure 6.8, it is clear that $h \equiv r \times S\,a$ and $b \equiv r \times C\,a$. Hence $r \equiv b \div C\,a$ and $h \equiv b \times (S\,a) \div C\,a$.



$$b \equiv r \times C\,a$$
$$h \equiv b \times (S\,a) \div C\,a \equiv b \times T\,a$$

Figure 6.12 Calculation of height

For example, if $a = 0.5$ radians (28.65°), and $b = 200$ feet, then

$$\begin{aligned} S\,a &\equiv \left(\frac{1}{2} - \frac{1}{48}\right) + \left(\frac{1}{3840} - \frac{1}{645120}\right) + \ldots \\ &\equiv .4794\ldots \\ C\,a &\equiv \left(1 - \frac{1}{8}\right) + \left(\frac{1}{384} - \frac{1}{46080}\right) + \ldots \\ &\equiv .8776\ldots \end{aligned}$$

and hence

$$h \equiv 200 \times \frac{.4794}{.8776} \equiv 109.2 \text{ feet, approximately}$$

This result could also be obtained by using the table of circular functions in Appendix B, in which case it would be necessary to convert the measure of the angle to degrees. It would also be more convenient to use the equivalent expression $h \equiv b \times T\,a$.

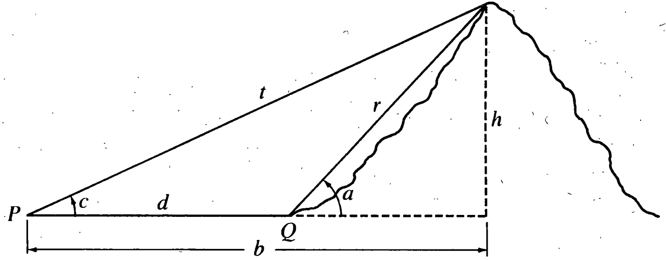


Figure 6.13 $h \equiv \dfrac{d \times (T\,a) \times T\,c}{(T\,a) - T\,c}$

Figure 6.13 illustrates the similar but slightly more complex problem of calculating the height of a mountain, where the distance to the base (that is, the horizontal distance to a point directly below the summit) cannot be measured directly. However, the distance from point P to a point Q (which is at the same level as P and directly between P and the vertical through the summit) can be measured, as can the angles to the summit from both P and Q. As shown in the figure, these measures are denoted by d (in feet) and by c and a (in radians), respectively.

It is clear from the figure that

$$\frac{h}{b} \equiv \frac{S\,c}{C\,c} \equiv T\,c \text{ and } \frac{h}{b-d} \equiv \frac{S\,a}{C\,a} \equiv T\,a$$

Therefore

$$b \equiv \frac{h}{T\,c} \text{ and } h \equiv (b - d) \times T\,a \equiv \left(\frac{h}{T\,c} - d\right) \times T\,a$$

Multiplying both sides by $T\,c$ and collecting terms in h gives

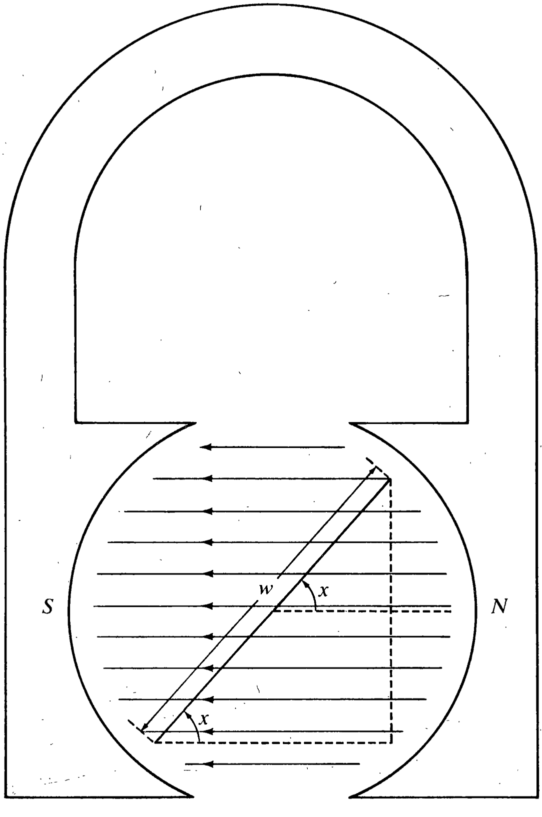
$$h \times (T\,c) - T\,a \equiv -d \times (T\,a) \times T\,c$$

Therefore

$$h \equiv \frac{d \times (T\,a) \times T\,c}{(T\,a) - T\,c}$$

(Do Exercises 6.25–6.26.)

Nongeometric applications. An electric generator produces a voltage by rotation of a coil in a magnetic field as illustrated in Figure 6.14. The amount of magnetic flux $F\,x$ passing through the coil



$F\,x \equiv s \times l \times w \times S\,x$

Figure 6.14 Magnetic flux through a generator coil when it is inclined at an angle x to the direction of the magnetic field is equal to $s \times l \times w \times S\, x$, where s is the field strength and where l is the length, w is the width, and $l \times w$ is the area enclosed by the coil. If the coil rotates at a constant angular velocity of v radians per second, then the angle x is a function of the time t:

$$x \equiv v \times t$$

Hence the magnetic flux through the coil is also a function of the time t:

$$F\, t \equiv s \times l \times w \times S\, v \times t$$

The voltage generated in the coil at any instant depends, however, not directly on the amount of flux through the coil but rather on the rate at which the flux is changing. Hence the voltage function $V\, t$ is the slope function of $F\, t$:

$$V\, t \equiv (D\, F)\, t$$

If the angular velocity v is equal to 1, then (from Equations 5.6 and 6.9)

$$\begin{aligned} V\, t \equiv (D\, F)\, t &\equiv (D\, (s \times l \times w) \times S)\, t \\ &\equiv (s \times l \times w) \times C\, t \end{aligned}$$

Hence the voltage is proportional to the cosine function. The graph of the cosine in Figure 6.5 shows that an alternating voltage is produced, the maximum voltage occurring when the coil is horizontal, and a change in direction of voltage occurring when the voltage passes through zero at angles of $\frac{\pi}{2}$ and $\frac{3}{2} \times \pi$ radians.

For a general value of the velocity v, the relation $(D\, S\, v \times)\, t \equiv v \times C\, v \times t$ (established in Exercise 6.27) can be applied to give the following general expression for the voltage:

$$V\, t \equiv v \times (s \times l \times w \times C\, v \times t)$$

This result shows that the voltage generated is proportional to the angular velocity of the coil.

(Do Exercises 6.27 – 6.32.)

The vibration of a mass freely suspended from a spring turns out to be a motion which is a sine function of the time t. The less familiar electrical vibrations or *oscillations* produced by connecting a coil to a charged condenser have the same form, and the reasons for the similarity become apparent in the equations describing the mechanical and the electrical vibrations. An example of each will now be examined.

Let a weight of mass m be suspended from a spring, and let $P\,t$ be the function which describes its position as measured from its equilibrium position (with positive direction downward), as shown in Figure 6.15. Since the force required to stretch a spring is proportional to the distance it is stretched, the force applied to the weight at time t is

$$F\,t \equiv -k \times P\,t$$

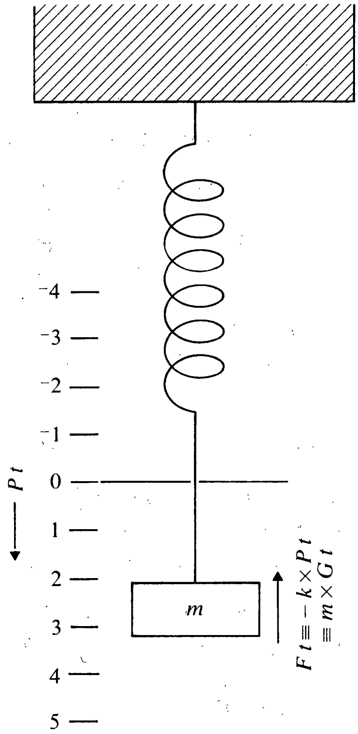


Figure 6.15 Position $(P\,t)$, accelerating force $(F\,t)$, and acceleration $(G\,t \equiv (D\,D\,P)\,t)$ for a vibrating body

The minus sign indicates that the direction of the force $F\,t$ is opposite to the direction of the displacement from the zero position.

But the acceleration of a mass m is related to the applied force $F\,t$ as follows:

$$F\,t \equiv m \times G\,t$$

Hence the acceleration $G\ t$ at time t is given by

$$G\ t \equiv \frac{F\ t}{m} \equiv -\frac{k}{m} \times P\ t$$

The velocity $V\ t$ is the rate of change of the position $P\ t$, that is, $V\ t \equiv (D\ P)\ t$. Since the acceleration is the rate of change of the velocity, then

$$G\ t \equiv (D\ V)\ t \equiv (D\ D\ P)\ t$$

Therefore

$$(D\ D\ P)\ t \equiv -\frac{k}{m} \times P\ t$$

If the mass and the stiffness of the spring (m and k) are chosen so that $\frac{k}{m} \equiv 1$, then $(D\ D\ P)\ t \equiv -P\ t$. Both the sine and the cosine functions satisfy this same relation and, more generally, if a and b are constants, then the function $P\ t \equiv (a \times \sin\ t) + b \times \cos t$ satisfies it as well. For

$$(D\ P)\ t \equiv (a \times \cos t) + b \times -\sin t$$

and

$$(D\ D\ P)\ t \equiv (a \times -\sin t) + b \times -\cos t \equiv -P\ t$$

The constants a and b are determined by the initial position and velocity of the mass at time $t = 0$. For example, if the mass is released from rest (that is, zero velocity) at an initial position i, then

$$P\ 0 \equiv (a \times \sin 0) + b \times \cos 0 \equiv i$$
$$(D\ P)\ 0 \equiv (a \times \cos 0) + b \times -\sin 0 \equiv 0$$

Therefore $b = i$ and $a = 0$. Finally

$$P\ t \equiv i \times \cos t$$

(Do Exercise 6.33.) The case of general values of k and m will be treated in the exercises.

The electrical circuit of Figure 6.16 contains a coil and a charged condenser (capacitor). The voltage $V\ t$, the current $J\ t$, and the amount of electrical charge in the condenser $Q\ t$ are all functions of time. They are related exactly as were the acceleration, velocity, and position functions of the preceding mechanical example. If l is the inductance of the coil in *henries*, and c is the capacity of the condenser in *farads*, then the voltage across the coil (which depends on the rate of change of the current $J\,t$) is given by $l \times (D\,J)\,t$, and the voltage across the condenser is $\frac{1}{c} \times Q\,t$.

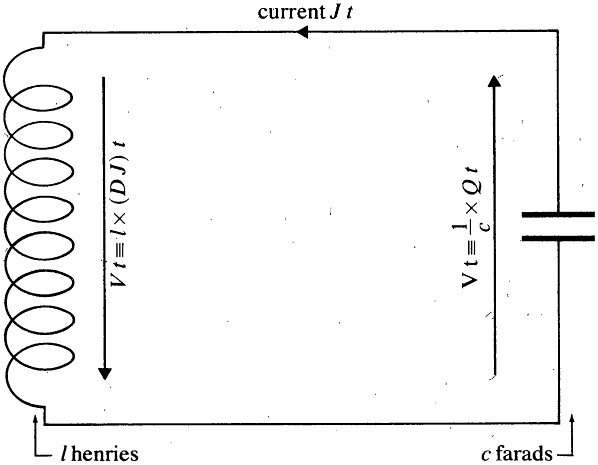


Figure 6.16 Relations between voltage $V\,t$, current $J\,t$, and charge $Q\,t$ in an oscillating circuit

Since the coil and condenser form a complete circuit, the sum of these voltages must be zero and therefore (since $(D\,J)\,t \equiv (D\,D\,Q)\,t$)

$$\frac{1}{c} \times Q\,t \equiv -l \times (D\,D\,Q)\,t$$

Hence

$$Q\,t \equiv -(c \times l) \times (D\,D\,Q)\,t$$

If $c \times l = 1$, the function $Q\,t$ clearly has the same form as the function $P\,t$:

$$Q\,t \equiv (a \times \sin t) + b \times \cos t$$

Therefore the charge Q (and hence the voltage across the condenser) oscillates in a manner analogous to the oscillations of a mass suspended on a spring. **(Do Exercises 6.34–6.36.)**

Exercises

6.1 (a) Determine the values of $\sin x$ and $\cos x$ for $x = -\frac{\pi}{2}$, $-\frac{\pi}{4}$, $0, \frac{\pi}{4}, \frac{\pi}{2}, 3 \times \frac{\pi}{4}$, and $7 \times \frac{\pi}{4}$.

(b) Use geometric methods to determine the values of $\sin \frac{\pi}{3}$ and $\cos \frac{\pi}{3}$.

(c) Determine the values of $\sin \frac{\pi}{6}$ and $\cos \frac{\pi}{6}$.

6.2 An angle whose measure lies at or between 0 and $\frac{\pi}{2}$ radians is said to be *in the first quadrant* or is said to be a *principal* angle. For each of the following angles, express the value of its sine in terms of the sine of an angle in the first quadrant.

(a) $3 \times \frac{\pi}{2}$ radians

(b) $-\frac{\pi}{2}$ radians

(c) $7 \times \frac{\pi}{4}$ radians

(d) $3 \times \frac{\pi}{4}$ radians

(e) $5 \times \frac{\pi}{4}$ radians

(f) $5 \times \frac{\pi}{6}$ radians

(g) $3 \times \pi$ radians

6.3 For each of the angles of Exercise 6.2, express

(i) the cosine in terms of the cosine of a principal angle
(ii) the sine in terms of the cosine of a principal angle
(iii) the cosine in terms of the sine of a principal angle

6.4 Write programs which will determine for any argument x (expressed in radians) the factor f and principal angle p such that $f = \pm 1$ and

(a) $\sin x \equiv f \times \sin p$

(b) $\cos x \equiv f \times \cos p$

(c) $\sin x \equiv f \times \cos p$

(d) $\cos x \equiv f \times \sin p$

Note that x may be negative or may exceed $2 \times \pi$.

6.5 Write programs which will determine for any argument x (expressed in radians) the factor f, the angle h, and the "function selector" s such that $0 \leq h \leq \frac{\pi}{4}$ and

(a) $\sin x \equiv ((s=0) \times f \times \cos h) + (s=1) \times f \times \sin h$
(b) $\cos x \equiv ((s=0) \times f \times \cos h) + (s=1) \times f \times \sin h$

6.6 Use the addition theorem for the sine function (Equation 6.5) to evaluate $\sin x + y$ for the following values of x and y, comparing each result with the known value of the sine of $x + y$:

(a) $x = \frac{\pi}{2}, y = \frac{\pi}{2}$
(b) $x = \pi, y = \frac{\pi}{2}$
(c) $x = \pi, y = \pi$
(d) $x = \frac{\pi}{4}, y = \frac{\pi}{4}$
(e) $x = \frac{\pi}{6}, y = \frac{\pi}{6}$
(f) $x = \frac{\pi}{3}, y = \frac{\pi}{3}$
(g) $x = \frac{\pi}{3}, y = \frac{\pi}{6}$
(h) $x = 2 \times \pi, y = a$

6.7 Repeat Exercise 6.6, substituting $\cos x + y$ for $\sin x + y$.

6.8 (a) Use Equation 6.1 and the addition theorem for the sine to show that

$$\sin x - y \equiv ((\cos y) \times \sin x) - (\sin y) \times \cos x$$

(b) Show that

$$\cos x - y \equiv ((\cos y) \times \cos x) + (\sin y) \times \sin x$$

6.9 Use the addition theorems for the sine and cosine to prove the following:

(a) $\sin 2 \times x \equiv 2 \times (\cos x) \times \sin x$
(b) $\cos 2 \times x \equiv (\cos x)^2 - (\sin x)^2$
(c) $\cos 2 \times x \equiv (2 \times (\cos x)^2) - 1$ (Use Equation 6.4.)
(d) $\cos 2 \times x \equiv 1 - 2 \times (\sin x)^2$
(e) $\cos y \div 2 \equiv \pm \surd\ (1 + \cos y) \div 2$ (Use the result of part (c) with $x = y \div 2$. The symbol ± means that one or the other sign is correct, not both.)
(f) $\sin y \div 2 \equiv \pm \surd\ (1 - \cos y) \div 2$ (Use the result of part (d) with $x = y \div 2$.)

6.10 For the following values of y, use parts (e) and (f) of Exercise 6.9 to determine the values of $\sin y \div 2$ and $\cos y \div 2$, and where possible compare the results with known values:

(a) $2 \times \pi$
(b) π
(c) $\frac{\pi}{2}$
(d) $\frac{\pi}{3}$
(e) $\frac{\pi}{4}$
(f) $\frac{\pi}{6}$

6.11 For each of the angles $a = 0, \frac{\pi}{4}, \frac{\pi}{2}, 3 \times \frac{\pi}{4}, \pi, 2 \times \pi$, and 2, make a table showing

(a) the slope of $\sin a$ as determined graphically from Figure 6.5 (by laying a ruler tangent to the graph of $\sin a$ and measuring its slope as accurately as possible)

(b) the slope of $\sin a$ as determined by Equation 6.11 and the graph of $\cos a$ in Figure 6.5

(c) the slope of $\cos a$ as determined graphically

(d) the slope of $\cos a$ as determined by Equation 6.11

6.12 Use the polynomials of Equation 5.14 to evaluate the functions $S\ x$ and $C\ x$ for $x = \frac{\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}$, and $-\frac{\pi}{6}$, comparing each result with the known values of $\sin x$ and $\cos x$.

6.13 In this exercise use the program of Exercise 5.33 (b) to evaluate the functions S and C. For the cases $x = 0, \frac{\pi}{2}, \frac{1}{2}$, and 1, compare the values of

(a) $S\ x$, $S\ x + 2 \times \pi$, and $S\ x - 2 \times \pi$ (See Equation 6.2.)

(b) $C\ x$, $C\ x + 2 \times \pi$, and $C\ x - 2 \times \pi$

(c) $C\ x$ and $C - x$ (See Equation 6.1.)

(d) $S\ x$ and $- S - x$

(e) $S\ x$ and $C\ \frac{\pi}{2} - x$ (See Equation 6.3.)

(f) $C\ x$ and $S\ x + \frac{\pi}{2}$

6.14 The evaluation of the polynomials for the sine and cosine (Equation 5.14) requires the use of a large number of terms if the argument is large. The programs required in Exercise 6.5 reexpress the functions $\sin x$ and $\cos x$ in terms of $\sin h$ and $\cos h$, where $0 \le h \le \frac{\pi}{4}$.

(a) Write an efficient program for the evaluation of $S\ x$ based on Exercise 6.5.

(b) Write an efficient program which evaluates $\cos x$ if $g = 0$, and $\sin x$ if $g = 1$.

6.15 Use the polynomials for $S\ x$ and $C\ x$ to compute the following values of the tangent function:

(a) $T \frac{\pi}{6}$

(b) $T \frac{\pi}{3}$

(c) $T\ 0$

(d) $T \frac{\pi}{2}$

6.16 Establish the following identity:

$$1 + (T\,x) \times T\,x \equiv 1 \div (C\,x) \times C\,x$$

6.17 Use the addition theorems for the sine and cosine to derive similar expressions for the following tangent functions. Check each result by applying it to at least one case for which the correct result is known (for example, for $x = \frac{\pi}{6}$, $T\,2 \times x \equiv T\frac{\pi}{3} \equiv \sqrt{}\,3$).

(a) $T\,x + y$ (c) $T\,2 \times x$

(b) $T\,x - y$ (d) $T\frac{x}{2}$

6.18 For angles with measures of $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ radians, make a table showing
(a) the corresponding number of degrees
(b) the sine as determined from Appendix B
(c) the cosine as determined from Appendix B
(d) the sine and cosine as determined in earlier exercises

6.19 (a) Use linear interpolation in the table of Appendix B to determine both the sine and the cosine of an angle of 1 radian.
(b) Use Equation 6.4 to check the result of part (a).
(c) Use the polynomials for $S\,x$ and $C\,x$ to check the result of part (a).

6.20 If $S\,x$, $C\,x$, $S\,a$, and $C\,a$ are known, then the addition theorems can be used repeatedly to obtain first $S\,x + a$ and $C\,x + a$, then $S\,x + 2 \times a$ and $C\,x + 2 \times a$, then $S\,x + 3 \times a$ and $C\,x + 3 \times a$, and so on.
(a) Compute $S\,0$, $C\,0$, $S\,.0175$, and $C\,.0175$ and use them with the addition theorems to compute the first few entries of a table of sines and cosines for an interval of .0175 radians. (NOTE: .0175 radians equals 1 degree.)
(b) Write a program for the method of part (a), including the calculation of $S\,.0175$ and $C\,.0175$.
(c) Execute the program of part (b) and compare the results with the tabulated values of Appendix B to observe the cumulative effects of round-off error.

6.21 Use the tables of Appendix B to determine the principal angle (in radians) whose
(a) sine is $\frac{1}{2}$
(b) cosine is $\frac{1}{2}$

(c) tangent is 1
(d) sine is .6 (use interpolation)
(e) cosine is .3 (use interpolation)

6.22 For a circle of unit radius, the perimeter of the circumscribed hexagon is clearly $12 \times T\frac{\pi}{6}$. Use the result of Exercise 6.17 (d) and the method of Figure 2.7 to develop a program for approximating π by the perimeters of circumscribed polygons.

6.23 (a) Write an expression for the altitude of a triangle in terms of the length of one side and the angle (in radians) which it makes with the base.
(b) Write an expression for the area of a parallelogram in terms of its sides and the smallest included angle.

6.24 Write a program to determine the angles in a right triangle whose legs have lengths x and y. (Use the polynomials for $S\,x$ and $C\,x$ and the method of successive approximation employed in Program 2.6.)

6.25 (a) Determine the height of a building if the line of sight to the top from a point 148 feet from the base makes an angle of 56° with the horizontal.
(b) Determine the distance of a point from the base of a building if the line of sight to the top makes an angle of 37° with the horizontal and the building is known to be 240 feet high.

6.26 Determine the height of the mountain of Figure 6.13 if $d = 4800$ feet, $c = 26°$, and $a = 39°$.

6.27 (a) As stated in the discussion of Figure 6.14, $(D\,S\,v\times)\,t \equiv v \times C\,v \times t$. Prove this result by determining the slope of the polynomial

$$S\,v \times t \equiv \left((v\times t) - \frac{(v\times t)^3}{!\,3}\right) + \left(\frac{(v\times t)^5}{!\,5} - \frac{(v\times t)^7}{!\,7}\right) + \ldots$$

obtained from Equation 5.14.
(b) Determine the slope of the function $C\,v \times t$.

6.28 If the coil in the generator of Figure 6.14 has n turns rather than 1, the voltage produced is increased by a factor of n and the expression for the voltage becomes

$$V\,t \equiv n \times v \times s \times l \times w \times C\,v \times t$$

where v is the angular velocity in radians per second, s is the field strength in webers per square meter, and l and w

are the coil dimensions in meters.

(a) At what angles do the maximum values of the voltage occur?

(b) What is the maximum voltage produced if $s = .08252$ webers per square meter, $v = 120 \times \pi$ radians per second, $l = .1$ meters, $w = .05$ meters, and $n = 1000$ turns?

(c) What is the voltage produced at the instant when $t = \frac{1}{480}$ seconds?

(d) For any periodic function of time, the *frequency* of the function (expressed in *cycles per second*) is defined as the number of periods occurring in one second. What is the frequency of the voltage produced by the generator?

6.29 Use the results of Exercise 6.27 and Exercise 6.9 to show that

(a) the function $\frac{x}{2} + \frac{S\,2 \times x}{4}$ has the slope function $(C\,x)^2$

(b) the function $\frac{x}{2} - \frac{S\,2 \times x}{4}$ has the slope function $(S\,x)^2$

6.30 Use the methods of Chapter 5 to determine

(a) an expression for the area to the left of the line $x = t$ and enclosed by the x-axis and the sine function $S\,x$ (The expression must be valid for $0 \leq t \leq \pi$.)

(b) an expression for the area enclosed by the x-axis, the cosine function $C\,x$, and the lines $x = 0$ and $x = t$

(c) the area enclosed by the x-axis and the first loop of the sine function (that is, from $x = 0$ to $x = \pi$)

6.31 Use the methods of Chapter 5 and the results of Exercise 6.29 to determine

(a) the area enclosed by the x-axis, the function $(S\,x) \times S\,x$, and the line $x = t$

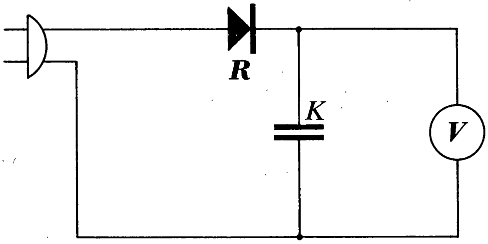
(b) the area enclosed by the x-axis and the first loop of the function $(S\,x) \times S\,x$

(c) the area enclosed by the x-axis and one loop of the function $(C\,x) \times C\,x$

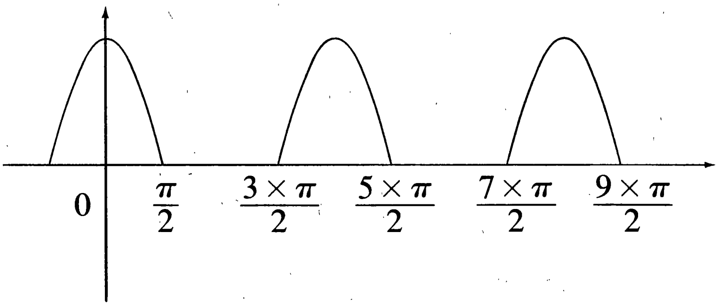
6.32 As shown in Figure 6.14, the voltage generated for and available in domestic wiring is a function of time t and has the form

$$V\,t \equiv m \times C\,2 \times \pi \times f \times t$$

where m is the maximum voltage and f is the frequency (in cycles per second) of the supply. The value of m can be determined by connecting a direct current voltmeter V, a rectifier R, and large condenser K (10 microfarads) as follows:



(The maximum voltage is observed because the rectifier permits current to pass in one direction only, and the condenser charges to the peak voltage and is discharged between peaks only very slightly by the current drawn by the meter.) If the condenser is replaced with a lamp, the meter records only the average voltage a, which is the average over cycles of the form shown below.



The zero voltage portions shown in the diagram are due to the prevention of reverse current by the rectifier.

(a) Find the ratio $\frac{a}{m}$ between the average and the maximum voltages. (Use the results of Exercise 6.30.)

(b) The average voltage a determined in part (a) is *not* 110 volts. The 110-volt figure is based on the square root of the average of the *squares* of the voltages and is therefore called the rms (root mean square) voltage. Using the results of Exercise 6.31, determine the relation between the rms voltage and the maximum m. (Since the power in watts delivered to a resistor R by a current $I=\frac{V}{R}$ is equal to the product $V\times I=\frac{V^2}{R}$, the rms voltage is a good indication of the *power* supplied by an alternating voltage.)

6.33 (a) Use the results of Exercise 6.27 to obtain an expression for the motion of the mass of Figure 6.15 which is valid for general values of k and m.

6.34 (a) Show that the frequency f of the electrical oscillation produced by the circuit of Figure 6.16 is given by $f=\frac{2\times\pi}{\sqrt{l\times c}}$.

(b) What is the frequency of the oscillation if $l=4$ millihenries (4×10^{-3} henries) and $c=.001$ microfarads ($.001\times 10^{-6}$ farads)?

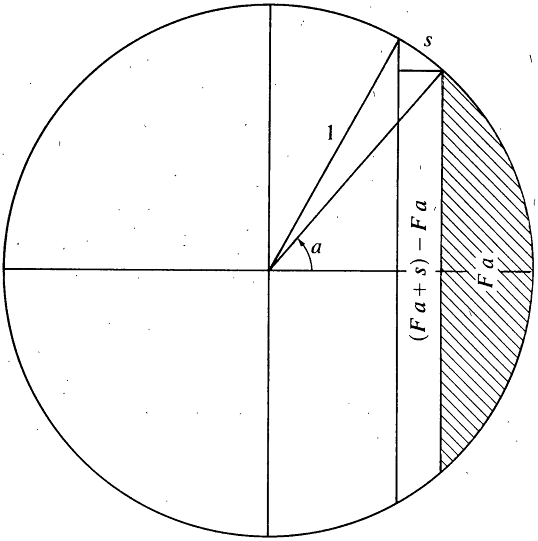
(c) If $c=.001$ microfarads, choose a value of l which will produce a signal for a radio station broadcasting at a frequency of 1 megacycle per second (10^6 cycles per second).

6.35 Let $F\,a$ be the area of the segment of a unit circle which is cut off by a chord that subtends an angle of $2\times a$ at the center of the circle. Then it is clear from the accompanying figure that $\frac{(F\,a+s)-F\,a}{s}$ is approximately equal to $\frac{(s\times S\,a)\times(2\times S\,a)}{s}\equiv 2\times(S\,a)\times S\,a$. Therefore $(D\,F)\,a$ $\equiv 2\times(S\,a)\times S\,a$. Use the results of Exercise 6.29 to determine

(a) an expression for $F\,a$

(b) the area of the unit circle

(c) the area of a circle of radius r, showing the entire derivation from a modified figure for the area



6.36 Consider a sphere of unit radius with center at the origin and a plane perpendicular to the x-axis intersecting the sphere at points for which $x = C\,a$. Thus each of the radii to the circle of intersection of the sphere and the plane makes an angle of a radians with the x-axis. Let $G\,a$ be the surface area of that portion of the sphere to the right of the plane.

(a) Draw a figure similar to that of Exercise 6.35 and use it to show that the slope of the function $G\,a$ is given by

$$(D\,G)\,a \equiv 2 \times \pi \times S\,a$$

(b) Determine the surface area of the zone of the sphere cut off by the plane.

(c) Determine the surface area of the entire sphere.

Chapter Seven

Inverse and Reciprocal Functions

Introduction

If G is any monadic function, then the function $1 \div G$ is called the *reciprocal* of G and is denoted by $\overline{G}$. Hence

$$(\overline{G}\,x) \times G\,x \equiv 1 \qquad (7.1)$$

For example, if $G\,x \equiv x \ast n$, then $\overline{G}\,x \equiv x \ast -n$, since $(x \ast -n) \times x \ast n \equiv x \ast 0 \equiv 1$. Similarly, $(\ast\, x) \times \ast -x \equiv 1$ (according to Exercise 5.19) and therefore $\overline{\ast}\, x \equiv \ast -x$. Moreover it is obvious that taking the reciprocal of a reciprocal yields the original function, that is, $\overline{\overline{G}}\,x \equiv G\,x$. Since the reciprocal $\overline{G}\,x$ becomes infinite at any point for which $G\,x$ is zero, the reciprocal function cannot be defined at all points. This is illustrated by the graphs of S and $\overline{S}$ in Figure 7.1.

If F is a function such that $F\,G\,x \equiv x$, then F is called the *inverse* of G and is denoted by G'. Hence $G'\,G\,x \equiv x$. For example, if $G\,x \equiv x \ast n$, then $G'\,x \equiv x \ast 1 \div n$, since $(x \ast n) \ast 1 \div n \equiv x \ast 1 \equiv x$.

In this example it is also true that $G\,G'\,x \equiv x$; since $(x \ast 1 \div n) \ast n \equiv x \ast 1 \equiv x$. In other words, in this example G is also the inverse of the function G'. This is true for any function G, that is, $G'\,G\,x \equiv x$ implies that $G\,G'\,x \equiv x$. For, applying the function G to both sides of $G'\,G\,x \equiv x$ yields $G\,G'\,G\,x \equiv G\,x$. Since $G\,x$ is some value y, this may be written as $G\,G'\,y \equiv y$. Finally

$$\left.\begin{aligned} G\,G'\,x \equiv x \\ G'\,G\,x \equiv x \end{aligned}\right\} \qquad (7.2)$$

The relation between the function $x \ast 3$ and its inverse is apparent from their graphs shown in Figure 7.2; one curve is obtained from the other by reflecting it in the 45° line. In other words, the graph

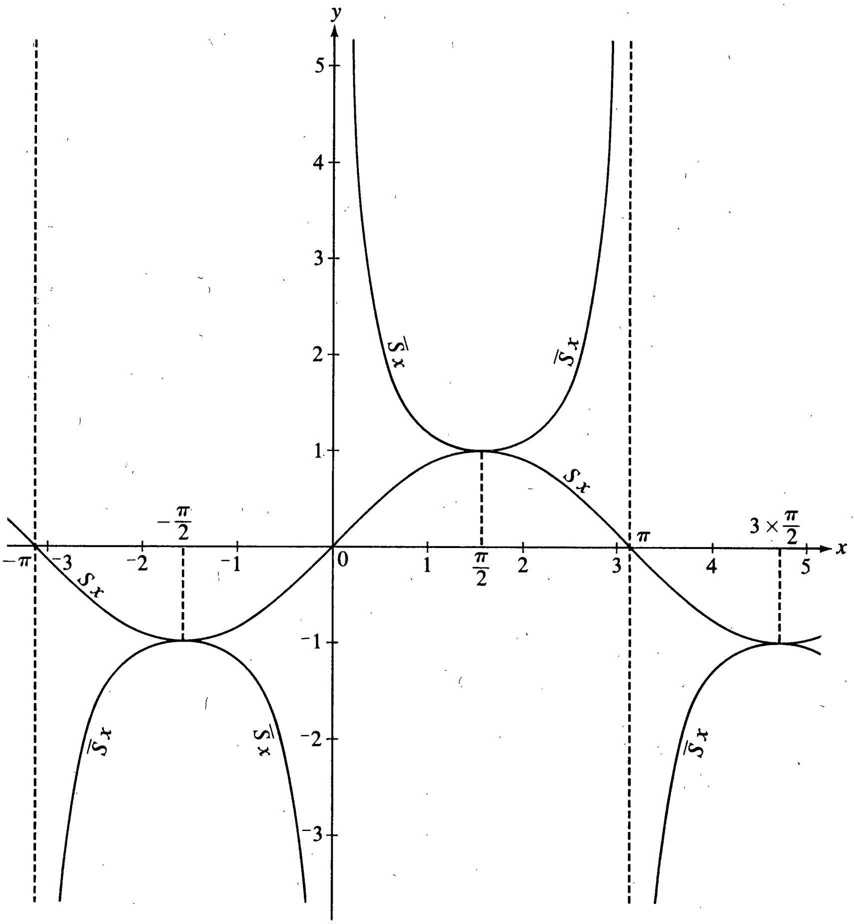


Figure 7.1 The sine (S) and cosecant ($\overline{S}$) functions

of the inverse is obtained by interchanging the x and y coordinate axes. This relation clearly holds between the graph of any function G and the graph of its inverse G'.

The graph of the inverse function S' of the sine function obtained in this manner from the graph of the sine function appears in Figure 7.3. It illustrates the fact that the inverse function may be multivalued; that is, for any one value of the argument there may be several suitable

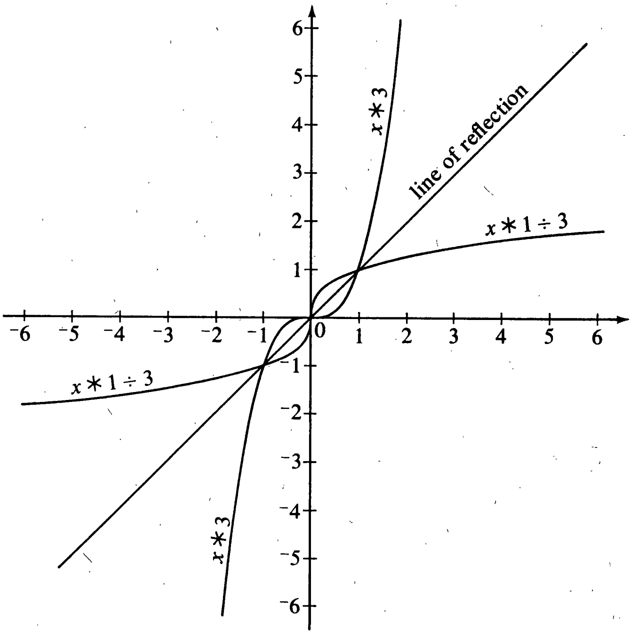


Figure 7.2 The function $x * 3$ and its inverse

values of the function. For example, if $x = 1$, then $S'\ x$ has the possible values $\frac{\pi}{2}, \frac{\pi}{2} + 2 \times \pi, \frac{\pi}{2} + 4 \times \pi$, and so forth. If $G'\ x$ is multivalued, some single-valued section of the graph is chosen as the definition of G'. In the case of the sine function, for example, the section AB $\left(\text{from } S'\ x = -\frac{\pi}{2} \text{ to } \frac{\pi}{2}\right)$ is used. In the case of a periodic function such as the sine, it is clear that other possible values of the inverse can be obtained by adding some multiple of the period to the value given by the chosen single-valued function.

(Do Exercises 7.1 – 7.3.)

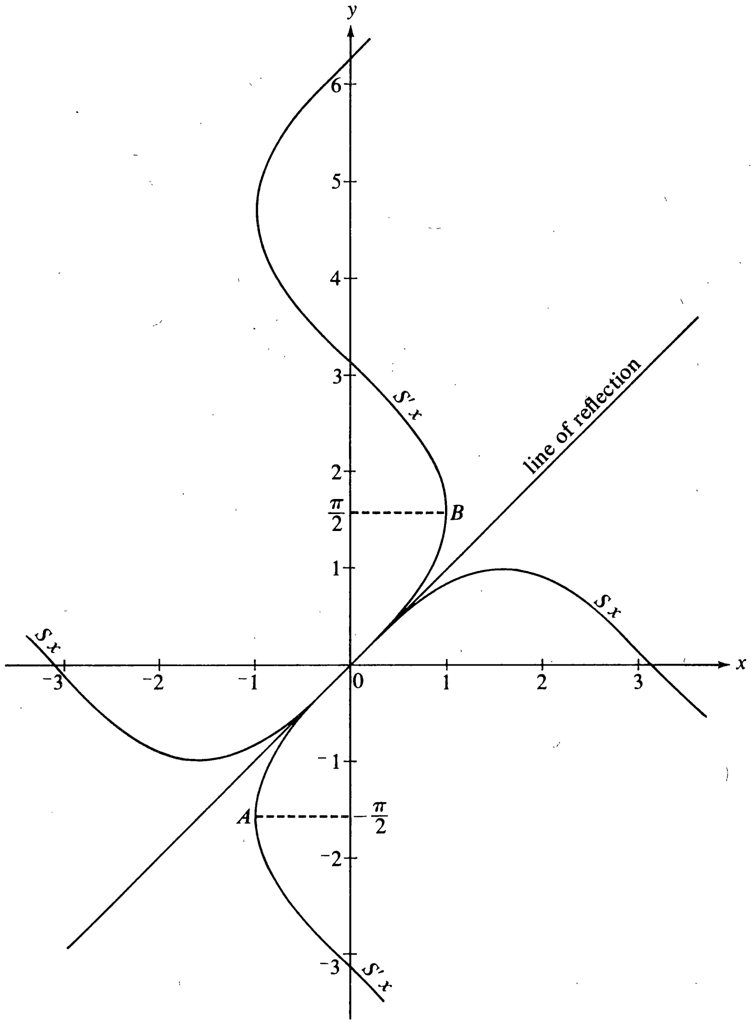


Figure 7.3 The sine (S) and arcsine (S') functions

The Slope of the Reciprocal $\overline{G}$

The slope of the function ($\overline{G}\ x$) × $G\ x$ occurring on the left side of Equation 7.1 can be obtained by applying the rule for products (Equation 5.6):

$$D\,\overline{G} \times G \equiv ((G) \times D\,\overline{G}) + (\overline{G}) \times D\,G$$

But the slope of the right side of Equation 7.1 is clearly zero; therefore

$$((G) \times D\,\overline{G}) + (\overline{G}) \times D\,G \equiv 0$$

Dividing through by $(G) \times \overline{G}$ yields $\frac{D\,\overline{G}}{\overline{G}} + \frac{D\,G}{G} \equiv 0$, or

$$\frac{D\,\overline{G}}{\overline{G}} \equiv -\frac{D\,G}{G} \qquad (7.3)$$

In other words, the slope of $\overline{G}$ bears the same ratio to $\overline{G}$ as does the slope of G to the function G, except that it is opposite in sign. This is reasonable, since any fractional increase in G must be compensated by an equal fractional decrease in $\overline{G}$ so that their product remains equal to 1.

Equation 7.3 can be transformed by simple algebra into the following useful forms:

$$D\,\overline{G} \equiv -(\overline{G}) \times \frac{D\,G}{G}$$

and

$$D\,\overline{G} \equiv -\frac{D\,G}{(G) \times G} \qquad (7.4)$$

For example, if $G\,x \equiv x \ast n$, then it follows from Equations 7.4 and 5.4 that

$$(D\,\overline{G})\,x \equiv -\frac{n \times x \ast n - 1}{(x \ast n) \times x \ast n}$$
$$\equiv (-n) \times x \ast (-n) - 1$$

Since $\overline{G}\,x \equiv x \ast -n$, then

$$D\,x \ast -n \equiv (-n) \times x \ast (-n) - 1$$

In other words, the rule for obtaining the slope of $x \ast n$ given in Equation 5.4 applies for negative values of the exponent as well. In particular, $D\frac{1}{x} \equiv D\,x \ast -1 \equiv -1 \times x \ast -2 \equiv -\frac{1}{x^2}$ and $D\,x \ast -2 \equiv -2 \times x \ast -3$.

(Do Exercises 7.4–7.6.)

Reciprocals of the Circular Functions

The reciprocal of the cosine function C is denoted by $\overline{C}$ and is called the *secant* function. It may be evaluated by using the polynomial for $C\,x$ and then taking the reciprocal. The slope of the secant can be obtained from Equations 7.4 and 6.10:

$$D\,\overline{C} \equiv \frac{-D\,C}{(C)\times C} \equiv \frac{S}{(C)\times C} \equiv (S)\times(\overline{C})\times\overline{C} \equiv \frac{T}{C} \equiv T\times\overline{C} \qquad (7.5)$$

The reciprocal of the sine function S is called the *cosecant.* Its slope is given by

$$D\,\overline{S} \equiv \frac{-C}{(S)\times S} \equiv -(C)\times(\overline{S})\times\overline{S} \equiv -(\overline{T})\times\overline{S} \qquad (7.6)$$

The function $\overline{T}$ occurring in Equation 7.6 is the reciprocal of the tangent function T and is called the *cotangent.* Since $\overline{T} \equiv (C) \div S \equiv \overline{S} \times C$, then

$$\begin{aligned} D\,\overline{T} &\equiv ((\overline{S})\times D\,C) + (C)\times D\,\overline{S} \\ &\equiv ((\overline{S})\times -S) + (C)\times -(\overline{T})\times\overline{S} \\ &\equiv -1 + (\overline{T})\times\overline{T} \\ &\equiv -1 + (\overline{T}) * 2 \end{aligned}$$

Similarly

(Do Exercises 7.7–7.9.)

$$\begin{aligned} T &\equiv \overline{C}\times S, \text{ and hence} \\ D\,T &\equiv ((\overline{C})\times D\,S) + S\times D\,\overline{C} \\ &\equiv 1 + (T) * 2 \end{aligned} \qquad (7.7)$$

The Slope of the Inverse G′

Equation 7.2 (that is, $G\,G'\,x \equiv x$) can be used to obtain the slope of G'. Taking the slopes of both sides yields

$$(D\,G\,G')\,x \equiv 1 \qquad (7.8)$$

However, the left side of this equation is the slope of a composite function of the form $G\,F\,x$ whose slope has not yet been determined.

The slope of the composite function $G\,F$ can be determined by the basic method presented in Chapter 5. The secant slope $D_s\,G\,F$ is given by

$$\begin{aligned} (D_s\,G\,F)\,x &\equiv \frac{(G\,F\,x+s) - G\,F\,x}{s} \\ &\equiv \frac{(G\,F\,x+s) - G\,F\,x}{(F\,x+s) - F\,x} \times \frac{(F\,x+s) - F\,x}{s} \end{aligned}$$

As s approaches 0, the second factor clearly approaches $(D\,F)\,x$; the first factor approaches $(D\,G)\,F\,x$, that is, the function $D\,G$ evaluated for the argument $F\,x$. For,

$$\frac{(G\,F\,x+s)-G\,F\,x}{(F\,x+s)-F\,x}\equiv(D_p\,G)\,F\,x$$

where $p\equiv(F\,x+s)-F\,x$. When s approaches 0, so does p. But as p approaches 0, the first factor approaches $(D\,G)\,F\,x$. Finally,

$$(D\,G\,F)\,x\equiv((D\,G)\,F\,x)\times(D\,F)\,x$$

or

$$D\,G\,F\equiv((D\,G)\,F)\times D\,F \qquad (7.9)$$

For example, if $G\,x\equiv x\ast 4$, and $F\,x\equiv x\ast 2$, then $(D\,F)\,x\equiv 2\times x$, $(D\,G)\,x\equiv 4\times x\ast 3$, and

$$\begin{aligned}(D\,G)\,F\,x&\equiv 4\times(F\,x)\ast 3\\&\equiv 4\times(x\ast 2)\ast 3\\&\equiv 4\times x\ast 6\end{aligned}$$

Hence

$$\begin{aligned}(D\,G\,F)\,x&\equiv((D\,G)\,F\,x)\times(D\,F)\,x\\&\equiv(4\times x\ast 6)\times 2\times x\\&\equiv 8\times x\ast 7\end{aligned}$$

This result can be corroborated by noting that $G\,F\,x\equiv(x\ast 2)\ast 4\equiv x\ast 8$ and taking the slope of this function directly.

The slope of G' can now be obtained from Equation 7.8 by using the result of Equation 7.9 with $F\equiv G'$:

$$(D\;G\;G')\;x\equiv((D\;G)\;G'\;x)\times(D\;G')\;x\equiv 1$$

Therefore

$$(D\;G')\;x\equiv\frac{1}{(D\;G)\;G'\;x} \qquad (7.10)$$

For example, if $G\,x\equiv x\ast 4$, then $G'\,x\equiv x\ast 1\div 4$ and

$$(D\,G')\,x\equiv\frac{1}{4\times(G'\,x)\ast 3}\equiv\frac{1}{4\times(x\ast 1\div 4)\ast 3}$$

Finally

$$\begin{aligned}D\,x\ast 1\div 4&\equiv\frac{1}{4\times x\ast 3\div 4}\\&\equiv\frac{1}{4}\times x\ast -3\div 4\end{aligned}$$

This result can be compared with the direct application of Equation 5.4. **(Do Exercises 7.10–7.12.)**

Inverses of the Circular Functions

The inverse functions of the circular functions are very useful, since they determine the arc associated with a given value of sine, cosine, or tangent. The inverse tangent function is the most easily derived, and S' and C' can be determined indirectly by first computing the corresponding tangent from one or other of the following relations:

$$T \equiv \frac{S}{\mathrm{C}} \equiv \frac{S}{\sqrt{1-(S)*2}} \equiv \frac{\sqrt{1-(C)*2}}{C} \equiv \sqrt{((\overline{C})*2)-1} \qquad (7.11)$$

The inverse sine, cosine, and tangent are frequently called the *arcsine*, *arccosine*, and *arctangent* respectively.

The slope of the inverse tangent T' is given by Equations 7.10 and 7.7:

$$(D\,T')\,x \equiv \frac{1}{(1+(T)*2)\;T'\,x} \equiv \frac{1}{1+(T\,T'\,x)*2} \equiv \frac{1}{1+x^2}$$

In words, $(D\,T')\,x$ is equal to 1 divided by the function $1+(T)*2$ evaluated at $T'\,x$ which, since $T\,T'\,x \equiv x$, is equal to $1 \div 1 + x^2$.

Since

$$\frac{1}{1+x^2} \equiv (1-x^2)+(x^4-x^6)+\ldots$$

(as can be verified by multiplying both sides by $1+x^2$), then

$$(D\,T')\,x \equiv (1-x^2)+(x^4-x^6)+(x^8-x^{10})+\ldots$$

Hence

$$T'\,x \equiv a+\left(x-\frac{x^3}{3}\right)+\left(\frac{x^5}{5}-\frac{x^7}{7}\right)+\ldots$$

as can be verified by taking the slope of the polynomial on the right. The constant a can be evaluated by observing that the angle having a tangent of 0 is itself 0 and hence $T'\,0 \equiv 0$. But the foregoing polynomial for $T'\,x$ shows that $T'\,0 \equiv a$. Hence $a \equiv 0$ and

$$T'\,x \equiv \left(x-\frac{x^3}{3}\right)+\left(\frac{x^5}{5}-\frac{x^7}{7}\right)+\ldots \qquad (7.12)$$

or

$$T'\,x \equiv x \times \left(1-\frac{x^2}{3}\right)+\left(\frac{x^4}{5}-\frac{x^6}{7}\right)+\ldots \qquad (7.13)$$

For example, the tangent of an angle of $\frac{\pi}{6}$ radians (30°) is $\frac{0.5}{\sqrt{0.75}}$ $\equiv\frac{1}{\sqrt{3}}$. Hence $T\frac{\pi}{6}\equiv\frac{1}{\sqrt{3}}$ and $\frac{\pi}{6}\equiv T'\frac{1}{\sqrt{3}}$. Finally, $\pi\equiv 6\times T'\frac{1}{\sqrt{3}}$ and Equation 7.13 yields

$$\pi = 6\times\frac{1}{\sqrt{3}}\times\left(1-\frac{1}{9}\right)+\left(\frac{1}{45}-\frac{1}{189}\right)+\left(\frac{1}{729}-\frac{1}{2673}\right)+\ldots$$

Since each parenthesized term in the above equation is positive, the sum of any finite number of terms is less than the exact value of π. The sums of the first few terms are shown in the second column of Table 7.4.

number of terms	lower bound	upper bound
1	3.0792	3.4641
2	3.1377	3.1561
3	3.1412	3.1416

Table 7.4 Approximations to $\pi = 6\times T'\ 1\div\sqrt{3}$

The terms in the equation can be regrouped as follows:

$$\pi = 6\times\frac{1}{\sqrt{3}}\times 1-\left(\frac{1}{9}-\frac{1}{45}\right)+\left(\frac{1}{189}-\frac{1}{729}\right)+\ldots$$

Again, each group is positive but is subtracted so that the sum of any finite number is *greater* than π. These sums are shown in the third column of Table 7.4. The last two entries of the table show that

$$3.1412<\pi<3.1416$$

The polynomial for $T'\ x$ given in Equation 7.12 can be evaluated only for values of the argument not exceeding 1 in absolute value, since for larger values of x the succeeding terms of the polynomial increase in value. Larger values of the argument can be handled by evaluating $T'\ 1\div x$ and using the relation

$$T'\ x\equiv\frac{\pi}{2}-T'\ 1\div x$$

developed in Figure 7.5. Equation 7.12 can be evaluated for the argument $1\div x$, for if $(|x)>1$ then $(|1\div x)<1$.

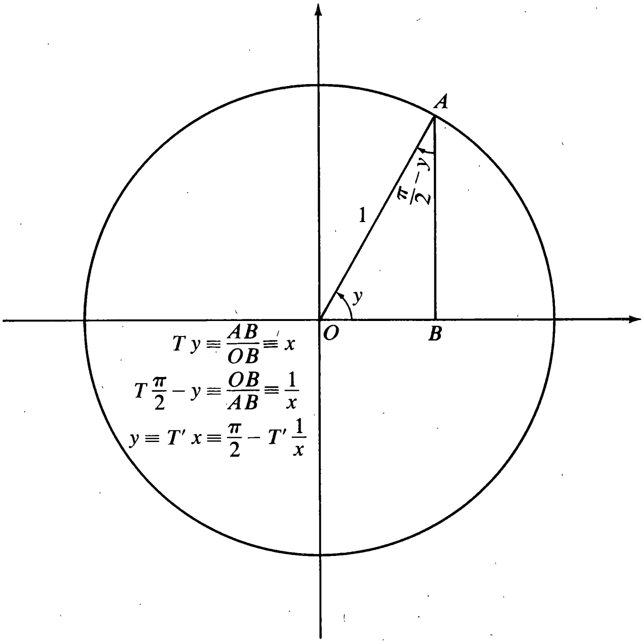


Figure 7.5 Reexpressing the inverse tangent for an argument in the range $^-1 \leq x \leq 1$

Since $T\,y$ is the quotient of an odd function divided by an even function, $T\,y$ is itself odd. Hence $T'\,x$ is odd. This is corroborated by its polynomial (Equation 7.12), which contains only odd powers of x. Equation 7.12 can also be written

$$T'\,x \equiv \left(0\,,1\,,0\,,\frac{^-1}{3}\,,0\,,\frac{1}{5}\,,0\,,\frac{^-1}{7}\,,0\,,\ldots\right)\Pi\,x \qquad (7.14)$$

The coefficients are easily remembered, since they differ from those for the sine only in that the denominators of the latter are factorials.

(Do Exercises 7.13–7.19.)

The Natural Logarithm

The *natural exponential* function

$$*\,x \equiv 1 + x + \frac{x^2}{!2} + \frac{x^3}{!3} + \ldots$$

defined by Equation 5.8 has an inverse denoted by $\ast'$ and called the *natural logarithm* function. According to Equation 7.10 the slope of the natural logarithm is given by

$$(D\ast')\,x \equiv \frac{1}{(D\ast)\,\ast' x}$$

Since $(D\ast) \equiv \ast$, then $(D\ast')\,x \equiv \frac{1}{\ast\ast' x}$. But $\ast\ast' x \equiv x$, and therefore

$$(D\ast')\,x \equiv \frac{1}{x} \tag{7.15}$$

In the case of the inverse tangent T', the expression for $D\,T'$ could be converted into a polynomial by division, that is,

$$\frac{1}{1+x^2} \equiv (1-x^2) + (x^4 - x^6) + \ldots$$

The same cannot be done for the expression of Equation 7.15, but can be done if some function $G\,x$ is first substituted for the argument x. Thus:

$$(D\ast')\,G\,x \equiv \frac{1}{G\,x} \tag{7.16}$$

If $G\,x \equiv 1-x$, then

$$\frac{1}{G\,x} \equiv \frac{1}{1-x} \equiv 1 + x + x^2 + x^3 + \ldots$$

as may be verified by multiplication. Therefore

$$(D\ast')\,G\,x \equiv 1 + x + x^2 + x^3 + \ldots$$

But from Equation 7.9:

$$\begin{aligned}(D\ast'\,G)\,x &\equiv (D\,G\,x) \times (D\ast')\,G\,x \\ &\equiv {}^{-}1 \times 1 + x + x^2 + x^3 + \ldots\end{aligned}$$

Therefore

$$\ast'\,G\,x \equiv \ast'\,1 - x \equiv a - x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

as can be verified by taking the slope of the polynomial on the right. The constant a can be shown to be 0 by noting that for $x = 0$ the value of the polynomial is a, and $\ast'\,1 - 0 \equiv \ast'\,1 \equiv 0$, since $\ast\,0 \equiv 1$. Therefore

$$\ast' 1 - x \equiv - x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \tag{7.17}$$

For example, if $x = 0.1$, then

$$\begin{aligned}\ast'(0.9) &\equiv - 0.1 + \frac{.01}{2} + \frac{.001}{3} + \frac{.0001}{4} + \ldots \\ &\equiv - 0.10535\ldots\end{aligned}$$

Moreover

$$\begin{aligned}\ast\ast' 0.9 &\equiv \ast - 0.10535\ldots \\ &\equiv (1 - .10535) + \left(\frac{.10535^2}{!2} - \frac{.10535^3}{!3}\right) + \ldots \\ &\equiv (1 - .10535) + (.00554 - .00019) + \ldots \\ &\equiv 0.90000\ldots\end{aligned}$$

(Do Exercise 7.20.) Hence $\ast\ast'$ (0.9) equals 0.9, as it should.

The polynomial of Equation 7.17 can be evaluated only for values of x having absolute values less than 1, since the later terms of the polynomial cannot be disregarded if the absolute value of x is greater than or equal to 1. A polynomial expression for the natural logarithm $\ast' n$ which can be evaluated for all positive values of the argument n will now be developed from Equation 7.17 and from the following property of the natural logarithm:

$$\ast' p \div q \equiv (\ast' p) - \ast' q \tag{7.18}$$

Equation 7.18 states that the natural logarithm of the quotient $p \div q$ is the natural logarithm of p less the natural logarithm of q. This result will not be proved until Chapter 8 (Equation 8.10 (g)), but is used here to unify the treatment of the natural logarithm.

Substituting $(-x)$ for x in Equation 7.17 yields

$$\ast'(1 + x) \equiv \left(x - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^4}{4}\right) + \ldots$$

Therefore

$$(\ast'(1 + x)) - \ast' 1 - x \equiv 2 \times \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

Applying Equation 7.18 for the case $p = 1 + x$ and $q = 1 - x$ yields

$$\ast'\left(\frac{1 + x}{1 - x}\right) \equiv 2 \times \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

Let $n = \frac{1+x}{1-x}$. Then $x = \frac{n-1}{n+1}$, as can be verified. Finally

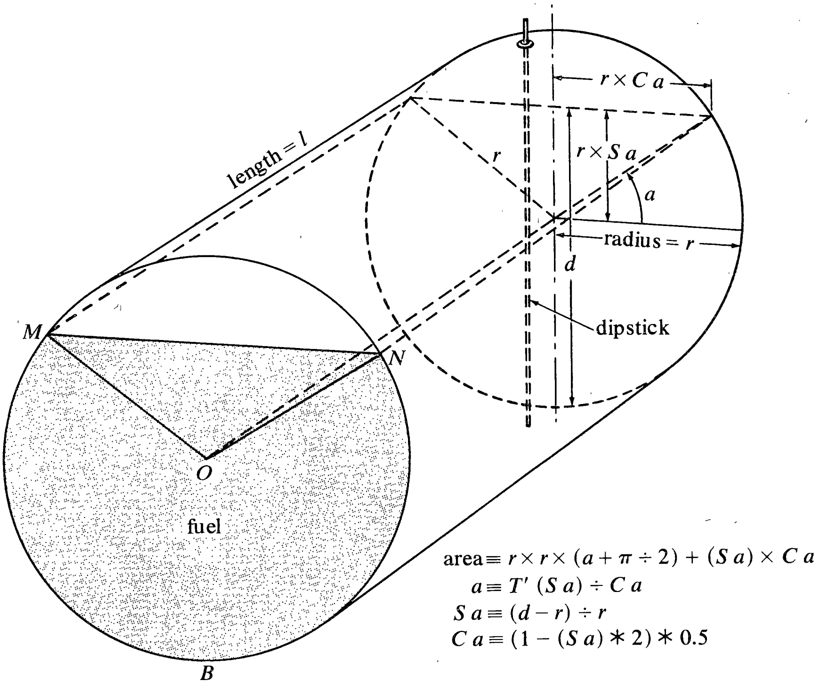
$$*'\, n \equiv 2 \times \left(\frac{n-1}{n+1} + \left(\frac{1}{3} \times \left(\frac{n-1}{n+1}\right)^3\right) + \left(\frac{1}{5} \times \left(\frac{n-1}{n+1}\right)^5\right) + \ldots\right) \qquad (7.19)$$

Since $\frac{n-1}{n+1}$ is less than 1 for all positive values of n, this polynomial can be evaluated for all positive values of the argument n. Moreover the values of the function $* x$ are positive for all values of x. Hence Equations 5.8 and 7.19 define $*$ and $*'$ as inverse functions, and each polynomial can be evaluated for any relevant argument.

(Do Exercises 7.21–7.26.)

Application of the Inverse Circular Functions

The need to calculate the angle corresponding to a given value of the sine, cosine, or tangent frequently arises. This will be illustrated for one case.



area $\equiv r \times r \times (a + \pi \div 2) + (S\,a) \times C\,a$
$a \equiv T'\ (S\,a) \div C\,a$
$S\,a \equiv (d - r) \div r$
$C\,a \equiv (1 - (S\,a) * 2) * 0.5$

Figure 7.6 Graduation of a dipstick

Consider the horizontal cylindrical fuel tank of Figure 7.6, in which a dipstick is used to determine the depth d of fuel in the tank. The dipstick is to be graduated to read directly in gallons. Assume that the length l, radius r, and depth d are all given in feet.

For specified values of l and r it is clear that the volume in cubic feet is determined by the depth d. In other words, the volume is a function of d. However, it is easier to determine the volume v as a function of the angle a, first determining a as a function of d. Thus $v = V\,a$ and $a = F\,d$. Hence, if g is the number of gallons, then

$$g \equiv 7.48 \times v \equiv 7.48 \times V\,a \equiv 7.48 \times V\,F\,d$$

Finally, the functions V and F must be determined.

The cross-sectional area of the fuel is the area of the sector $MONB$ plus the area of the triangle MON. Hence

$$\begin{aligned} V\,a &\equiv r \times r \times \left(a + \frac{\pi}{2}\right) + (S\,a) \times C\,a \\ a &\equiv T'\,(S\,a) \div C\,a \\ S\,a &\equiv (d - r) \div r \\ C\,a &\equiv (1 - (S\,a) \ast 2) \ast 0.5 \end{aligned}$$

This analysis is also valid for $d < r$, since a and $S\,a$ are then negative.

(Do Exercises 7.27–7.33.)

Exercises

7.1 For each of the following functions, write the simplest possible expression for the reciprocal function $\overline{F}\,x$:

(a) $F\,x \equiv x \ast 3$

(b) $F\,x \equiv x \ast n$

(c) $F\,x \equiv (x \ast m) \ast -n$

(d) $F\,x \equiv (\mathbf{c}\,\Pi\,x) \div \mathbf{d}\,\Pi\,x$

(e) $F\,x \equiv ((x + a) \ast 2) - (x - a) \ast 2$

7.2 For each of the following functions, write the simplest possible expression for the inverse function $F'\,x$:

(a) $F\,x \equiv x \ast 3$

(b) $F\,x \equiv x \ast n$

(c) $F\,x \equiv (x \ast m) \ast -n$

(d) $F\,x \equiv c \times x$

(e) $F\,x \equiv c + x$

7.3 For the case $x = 2$, $m = 3$, $n = 4$, and $c = 5$, check each of the results of Exercise 7.2 to show that

(i) $F'\,F\,x \equiv x$

(ii) $F\,F'\,x \equiv x$

7.4 Determine the slope functions $D\,F$ and $D\,\overline{F}$ for each of the functions in parts (a), (b), (c), and (e) of Exercise 7.1.

7.5 Use the fact that $F \div G \equiv F \times \overline{G}$ to show that

$$D\,F \div G \equiv \frac{((G)\times D\,F) - (F)\times D\,G}{(G)\times G}$$

7.6 For each of the following pairs of functions F and G, use the result of Exercise 7.5 to determine the slope of the function $(F) \div G$. Where possible, check the result by putting the function $(F) \div G$ in another form. For example, in part (a), $(F\,x) \div G\,x \equiv x \ast 3$.

(a) $F\,x \equiv x \ast 5, G\,x \equiv x \ast 2$
(b) $F\,x \equiv x \ast 2, G\,x \equiv x \ast 5$
(c) $F\,x \equiv x \ast m, G\,x \equiv x \ast n$
(d) $F\,x \equiv \ast\, x, G\,x \equiv \ast\, x$
(e) $F\,x \equiv \ast\, x, G\,x \equiv \ast -x$
(f) $F\,x \equiv a + x, G\,x \equiv b + x$
(g) $F\,x \equiv (a+x) \ast m, G\,x \equiv (b+x) \ast n$
(h) $F\,x \equiv S\,x, G\,x \equiv C\,x$
(i) $F\,x \equiv \mathbf{c}\,\Pi\,x, G\,x \equiv \mathbf{d}\,\Pi\,x$

7.7 Use the addition theorems for the sine and cosine to establish the following addition theorems for the tangent and cotangent:

(a) $T\,x+y \equiv \dfrac{(T\,x) + T\,y}{1 - (T\,x) \times T\,y}$

(b) $\overline{T}\,x+y \equiv \dfrac{((\overline{T}\,x) \times \overline{T}\,y) - 1}{(\overline{T}\,x) + \overline{T}\,y}$

7.8 Use the results of Exercise 7.7 to obtain the following slope functions directly from the limiting value of the secant slope (Equation 5.1):

(a) the tangent function T (Compare the result with Equation 7.7.)
(b) the cotangent function $\overline{T}$ (Compare with the text.)

7.9 Prove that $T\,x \equiv \overline{T}\,\frac{\pi}{2} - x$.

7.10 Use the rule for determining the slope of the function $G\,F$ in order to determine the slope of the function $H \equiv G\,F$ for each of the following pairs of functions. In each case compare the result with the slope obtained by applying earlier methods to some equivalent expression for the function H.

(a) $F\,x \equiv x \ast 2;\ G\,x \equiv x \ast 3$
(b) $F\,x \equiv x \ast 3;\ G\,x \equiv x \ast 2$
(c) $F\,x \equiv x \ast 4;\ G\,x \equiv 3 \times x$
(d) $F\,x \equiv 3 \times x;\ G\,x \equiv x \ast 4$
(e) $F\,x \equiv x \ast 4;\ G\,x \equiv -\,x$
(f) $F\,x \equiv -\,x;\quad G\,x \equiv x \ast 4$
(g) $F\,x \equiv x \ast 4;\ G\,x \equiv 3 + x$
(h) $F\,x \equiv 3 + x;\ G\,x \equiv x \ast 4$

7.11 Determine the slope function $D\,F$ for each of the following functions:
(a) $F\,x \equiv (6 + x \ast 3) \ast 2$
(b) $F\,x \equiv x \ast 1 \div 6$
(c) $F\,x \equiv ((3\,,1\,,2)\;\Pi\;x) \div (4\,,1\,,0\,,2)\;\Pi\;x$

7.12 Use Equation 7.9 to derive the slope of the function H from the first expression given for it, and compare the result with the slope as derived from the second expression.
(a) $H\,x \equiv S\,2 \times x \equiv 2 \times (C\,x) \times S\,x$
(b) $H\,x \equiv C\,2 \times x \equiv ((C\,x) \ast 2) - (S\,x) \ast 2$
(c) $H\,x \equiv S\,x \div 2 \equiv ((1 - C\,x) \div 2) \ast 1 \div 2$

7.13 Show that $T'\,x \equiv \frac{\pi}{2} - T'\,\frac{1}{x}$.

7.14 Let $Q\,n$ be the sum of the first n (nonzero) terms of the polynomial of Equation 7.12. Show that if $(|\,x) \le 1$, then
(a) $((Q\,n) - T'\,x) \ge 0$ for $n = 1, 3, 5, 7, \ldots$
(b) $((Q\,n) - T'\,x) \le 0$ for $n = 2, 4, 6, 8, \ldots$
(c) $((Q\,n) - Q\,n + 2) \ge 0$ for $n = 1, 3, 5, 7, \ldots$
(d) $((Q\,n) - Q\,n + 2) \le 0$ for $n = 2, 4, 6, 8, \ldots$
(e) the absolute value of the difference between $Q\,n$ and T' does not exceed the absolute value of the last term used in $Q\,n$. (HINT: Use the observations made in the evaluation of $T'\,1 \div \surd\,3$.)

7.15 Write a program to compute the inverse tangent $T'\,x$ to a specified tolerance t, using the polynomial of Equation 7.12 and the results of Exercises 7.13 and 7.14.

7.16 Verify that the functions T and T' as defined by Equations 5.14, 6.13, and 7.12 are in fact inverse for the following values of the argument x. (Compute both $T\,T'\,x$ and $T'\,T\,x$.)
(a) $x = 0$ (c) $x = \pi \div 6$
(b) $x = 0.1$ (d) $x = 1 \div \surd\,3$

7.17 Write programs to compute
(a) $S'\,y$ (b) $\overline{S}'\,y$ (c) $C'\,y$

7.18 Prove that

(a) $\overline{C}\,2\times x\equiv\dfrac{(\overline{C}\,x)\ast 2}{1-(T\,x)\ast 2}$

(b) $\overline{S}\,2\times x\equiv\dfrac{(\overline{S}\,x)\ast 2}{2\times\overline{T}\,x}$

(c) $\overline{T}\,2\times x\equiv\dfrac{((\overline{T}\,x)\ast 2)-1}{2\times\overline{T}\,x}$

7.19 Write a single program which computes to within a tolerance x_5 one of twelve functions of x_1 as determined by the arguments x_2 (type), x_3 (reciprocal), and x_4 (inverse), as follows:

1. The basic function involved is sine, tangent, or cosine, according to whether $x_2={}^-1$, 0, or 1.
2. The function selected in part 1 or its reciprocal is employed according to whether $x_3=1$ or ${}^-1$.
3. The function determined by parts (1) and (2) or its inverse is calculated according to whether $x_4=0$ or 1.

7.20 (a) Use Equation 7.17 to compute the natural logarithm

$\ast'\,y$ for $y=\frac{3}{4},\frac{1}{2}$, and $\frac{1}{4}$.

(b) Use Equation 5.8 to check the results of part (a).

7.21 Use the results of Exercise 7.20 (a) to test Equation 7.18 as thoroughly as possible. For example, $\ast'\left(\frac{1}{4}\div\frac{1}{2}\right)\equiv\ast'\left(\frac{1}{2}\right)$ should equal $\left(\ast'\frac{1}{4}\right)-\ast'\frac{1}{2}$, and $\ast'\left(\frac{1}{2}\div\frac{1}{4}\right)\equiv\ast'\,2$ can be checked by computing $\ast\ast'\,2$.

7.22 Use Equation 7.19 to evaluate $\ast'\,n$ for $n=\frac{1}{4},\frac{1}{2}$, and $\frac{3}{4}$, and compare the results with the results of Exercise 7.20 (a).

7.23 (a) Use Equation 7.19 to evaluate $\ast'\,n$ for $n=1, 2$, and 3.
(b) Use Equation 5.8 to check the results of part (a).

7.24 Since the terms of Equation 7.19 decrease slowly for large values of n, a great deal of calculation is required to achieve a few decimal places of accuracy in the approximation to $\ast'\,n$.

(a) Evaluate $\ast'\,4$ by the direct use of Equation 7.19, and then evaluate it by using Equation 7.18 in the following form:

$$\ast' 4 \equiv \ast' \left(2 \div \frac{1}{2}\right) \equiv (\ast' 2) - \ast' \frac{1}{2}$$

(b) Evaluate $\ast' 8 \equiv (\ast' 4) - \ast' \frac{1}{2}$.

(c) Evaluate $\ast' 10 \equiv \left(\ast' \frac{5}{4}\right) - \ast' \frac{1}{4}$.

7.25 (a) Use Equation 7.18 to show that $\ast' 1 \div q \equiv - \ast' q$.

(b) Show that Equation 7.19 yields the same result.

7.26 Write a program based on Equation 7.19 to compute $\ast' n$ to within a tolerance t.

7.27 Write a program to graduate the dipstick of Figure 7.6, that is, to determine for each foot of the dipstick the corresponding number of gallons of oil in the tank.

7.28 The functions A and B defined by Equations 5.11 and 5.12 can be derived from the hyperbola in a manner analogous to the derivation of the cosine and sine functions from the circle.† The functions A and B are therefore called the *hyperbolic cosine* and the *hyperbolic sine* respectively. The *hyperbolic tangent* (to be denoted by U) is defined as the analogue of the tangent, that is, $U \equiv (B) \div A$.

(a) Show that $((A\,x) \times A\,x) - (B\,x) \times B\,x \equiv 1$. (Use the method of proof employed for Equation 5.17.)

(b) Show that $1 - (U\,x) \times U\,x \equiv 1 \div (A\,x) \times A\,x$.

(c) Show that $(D\,U')\,x \equiv \frac{1}{1 - x^2}$. (Use the method employed for the inverse tangent.)

(d) Derive a polynomial expression for $U'\,x$.

(e) Evaluate $U'\,U\,x$ and $U\,U'\,x$ for several values of x so as to check the result of part (d).

(f) Write programs to determine $U'\,x$, $A'\,x$, and $B'\,x$.

7.29 Derive the following expressions for the slopes of the functions S', C', A', and B':

(a) $(D\,S')\,x \equiv \dfrac{1}{\sqrt{1 - x \ast 2}}$

(b) $(D\,C')\,x \equiv \dfrac{^{-}1}{\sqrt{1 - x \ast 2}}$

(c) $(D\,A')\,x \equiv \dfrac{1}{\sqrt{(x \ast 2) - 1}}$

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†See, for example, H. W. Reddick and F. H. Miller, *Advanced Mathematics for Engineers* (Wiley, 1938) or W. L. Hart, *Analytic Geometry and Calculus,* 2nd ed. (Heath, 1963).

(d) $(D\,B')\,x \equiv \dfrac{1}{\surd\,(x \ast 2) + 1}$

7.30 Show that if $H\,x \equiv \ast'\,G\,x$, then $(D\,H)\,x \equiv \dfrac{(D\,G)\,x}{G\,x}$.

7.31 Use the result of Exercise 7.30 to prove the following identities:

(a) $(D \ast'\,S)\,x \equiv \overline{T}\,x$

(b) $(D \ast'\,C)\,x \equiv -\,T\,x$

7.32 (a) Use the result of Exercise 7.31 (b) to derive an expression for the area enclosed by the x-axis, the graph of the function $T\,x$, and the vertical line $x = b$, for $0 \leq b < \frac{\pi}{2}$.

(b) Compute the value of the area for the case $b = \frac{\pi}{3}$.

Chapter Eight

The Exponential Function and Its Inverse

Introduction

Any dyadic function F gives rise to two different monadic functions if one or the other of the arguments is treated as a constant. Thus $p\ F\ q$ can be treated either as the monadic function $(p\ F)$ with the argument q or as the monadic function $(F\ q)$ with the argument p:

$$\begin{aligned} p\ F\ q &\equiv (p\ F)\ q \\ &\equiv p\ (F\ q) \end{aligned}$$

If F is commutative, the two cases give rise to functions of the same form. The noncommutative exponential function $x \ast n$ has been treated as the monadic function $x\ (\ast\ n)$, for example, in developing the binomial theorem (Equation 4.1). This chapter will treat the other case, $(x \ast)\ n$, as a monadic function of n. This function and its inverse are of great importance in mathematics.

As with the polynomial function and the circular functions, the treatment will proceed by first deriving an addition theorem and then using the theorem to derive the slope function. Since the symbol x has so far been used to denote the argument of the monadic function under study, the form $x \ast n$ will be discarded in favor of the form

$$b \ast x$$

where b is the *base* and $(b \ast)$ is the monadic function applied to the argument x. The function $b \ast$ is called the *base-b exponential* function, and its inverse $(b \ast)'$ is called the *base-b logarithm.*

The Base-b Exponential $(b*)\,x$

The function $b * x$ was originally defined only for integral values of x, and it denoted a product of x factors each equal to the base b. In this case it is clear that

$$b * x + y \equiv (b * x) \times b * y \qquad (8.1)$$

The function is extended to nonintegral values of x by simply requiring the foregoing addition theorem to hold for all values of x and y. For example, if $x = y = .5$, then

$$b \equiv b * 1 \equiv b * .5 + .5 \equiv (b * .5) \times b * .5$$

In other words, $b * .5$ is the square root of b. Consequently, Equation 8.1 is the addition theorem for the exponential.

The slope function $D\,b*$ is then obtained from the secant slope:

$$\begin{aligned}(D_s\,b*)\,x &\equiv \frac{(b * x + s) - b * x}{s}\\ &\equiv \frac{((b * x) \times b * s) - b * x}{s}\\ &\equiv \frac{(b * s) - 1}{s} \times b * x\end{aligned}$$

Thus the secant slope of $(b*)\,x$ is equal to the function $(b*)\,x$ itself multiplied by a factor which depends only on the horizontal interval s between the points of intersection of the secant and the graph of $(b*)\,x$. Table 8.1 shows the behavior of the factor $\frac{(b * s) - 1}{s}$ for various values of b. The entries in each column of the table appear to be approaching a limiting value. The slope function itself is therefore of the form

$$(D\,b*)\,x \equiv r \times b * x \qquad (8.2)$$

where r is the constant obtained (for a fixed value of b) as the limiting value of the ratio $\frac{(b * s) - 1}{s}$ as s approaches zero.

b s	$1 \div 3$	$1 \div 2$	1	2	3	4
1	¯0.6667	¯.5000	0	1.0000	2.0000	3.0000
$1 \div 2$	¯0.8453	¯.5858	0	.8284	1.4641	2.0000
$1 \div 4$	¯0.9607	¯.6364	0	.7568	1.2643	1.6569
$1 \div 8$	¯1.0265	¯.6640	0	.7241	1.1776	1.5137
$1 \div 16$	¯1.0617	¯.6783	0	.7084	1.1372	1.4481
$1 \div 32$	¯1.0800	¯.6857	0	.7007	1.1177	1.4168
$1 \div 64$	¯1.0892	¯.6894	0	.6969	1.1081	1.4014
$1 \div 128$	¯1.0939	¯.6913	0	.6950	1.1033	1.3938
$1 \div 256$	¯1.0963	¯.6922	0	.6941	1.1010	1.3901
$1 \div 512$	¯1.0974	¯.6927	0	.6936	1.0998	1.3882

Table 8.1 Behavior of $\frac{(b \ast s) - 1}{s}$

But, according to Equation 5.10, the function

$$(\ast\, r\times)\, x \equiv 1 + (r\times x) + \frac{(r\times x)^2}{!2} + \frac{(r\times x)^3}{!3} + \ldots$$

has the same property:

$$(D\ast\, r\times)\, x \equiv r\times (\ast\, r\times x) \tag{5.10}$$

Moreover $(\ast\, r\times x)$ is equal to $b\ast x$ for $x = 0$, since $(\ast\, r\times 0) \equiv 1$ and $(b\ast 0) \equiv 1$. It therefore appears that the function $\ast\, r\times x$ is identical with the function $b\ast x$ if a suitable value is chosen for r.

A value of r must be chosen such that $(b\ast x) \equiv \ast\, r\times x$. For $x = 1$ this becomes $b\ast 1 \equiv \ast\, r\times 1$, or (since $b\ast 1 \equiv b$) $b \equiv \ast\, r$. Therefore $r \equiv \ast'\, b$ and, finally,

$$b\ast x \equiv \ast\, x\times \ast'\, b \tag{8.3}$$

The function $\ast'$ is the natural logarithm, which can be evaluated by the polynomial of Equation 7.19. For example, to evaluate 0.9 $\ast$ 1.5, one can first show that $\ast'\, 0.9 \equiv {}^{-}0.10535\ldots$ (as in the example following Equation 7.17) and then substitute this result in Equation 8.3; this yields

$$\begin{aligned} 0.9 \ast 1.5 &\equiv \ast\, 1.5 \times {}^{-}0.10535 \equiv \ast\, {}^{-}0.15802 \\ &\equiv (1 - .15802) + \left(\frac{.15802^2}{2} - \frac{.15802^3}{3}\right) + \ldots \\ &\equiv .8539 \ldots \end{aligned}$$

This result can be checked as follows: Since $(0.9 \ast 1.5)^2 = (0.9)^3$, then $(0.9)^3$ should agree with $(.8539)^2$.

By applying the function $\ast'$ to both sides of Equation 8.3 and using the fact that $\ast' \ast z \equiv z$, one obtains the equivalent expression

$$\ast' \, b \ast x \equiv x \times \ast' \, b \qquad (8.4)$$

The exponential function satisfies one further important identity:

$$b \ast x \times y \equiv (b \ast x) \ast y \qquad (8.5)$$

This identity may be more familiar in the form $b^{x \times y} \equiv (b^x)^y$. Its proof for integral values of x and y is obvious. For arbitrary values of x and y it can be derived by applying Equations 8.3 and 8.4 as follows:

$$\begin{aligned} (b \ast x) \ast y &\equiv \ast\, y \times \ast' \, b \ast x \\ &\equiv \ast\, y \times x \times \ast' \, b \\ &\equiv \ast\, (x \times y) \times \ast' \, b \\ &\equiv b \ast x \times y \end{aligned}$$

(Do Exercises 8.1–8.3.)

The Base-b Logarithm $(b \ast)' \, x$

The base-b logarithm can be expressed in terms of the natural logarithm $\ast'$ as follows:

$$(b \ast)' \, x \equiv \frac{\ast' \, x}{\ast' \, b} \qquad (8.6)$$

To prove Equation 8.6 it is necessary to show that the function given for $(b \ast)'$ is in fact inverse to $b \ast$. In other words, it must be shown that the function on the right-hand side of Equation 8.6 yields the result x when applied to the argument $b \ast x$. But according to Equation 8.4,

$$\frac{\ast' \, (b \ast x)}{\ast' \, b} \equiv \frac{x \times \ast' \, b}{\ast' \, b} \equiv x$$

For example, if $b = 2$ and $x = 3$, then the base-2 logarithm of 3 is given by

$$(2 \ast)' \, 3 \equiv \frac{\ast' \, 3}{\ast' \, 2}$$

Both numerator and denominator can be evaluated by applying Equation 7.19:

$$\frac{\ast' 3}{\ast' 2} \equiv \frac{1.0985\ldots}{.6931\ldots} \equiv 1.585\ldots$$

This result can be checked by computing $2 \ast 1.585$ and comparing it with 3:

$$\begin{aligned} 2 \ast 1.585 &\equiv \ast\, (\ast' 2) \times 1.585 \\ &\equiv \ast\, .6931 \times 1.585 \\ &\equiv \ast\, 1.0985 \\ &\equiv 1 + 1.0985 + \frac{(1.0985)^2}{!2} + \frac{(1.0985)^3}{!3} + \ldots. \\ &\equiv 2.99\ldots \end{aligned}$$

(Do Exercises 8.4–8.5.)

Properties of $(b \ast\,)$ *and* $(b \ast\,)'$

The main properties of the base-*b* exponential and the base-*b* logarithm can now be derived rather easily. They will be collected in two groups, the first group arising from the addition theorem for the exponential (Equation 8.1) and the second group arising from the multiplication theorem of Equation 8.5:

$$\left.\begin{array}{rll} b \ast x + y \equiv & (b \ast x) \times b \ast y & \text{(a)} \\ 1 \equiv & (b \ast x) \times b \ast -x & \text{(b)} \\ b \ast x - y \equiv & (b \ast x) \div b \ast y & \text{(c)} \\ b \ast ((b \ast)' p) + (b \ast)' q \equiv & p \times q & \text{(d)} \\ b \ast ((b \ast)' p) - (b \ast)' q \equiv & p \div q & \text{(e)} \\ ((b \ast)' p) + (b \ast)' q \equiv & (b \ast)' p \times q & \text{(f)} \\ ((b \ast)' p) - (b \ast)' q \equiv & (b \ast)' p \div q & \text{(g)} \\ - (b \ast)' q \equiv & (b \ast)' 1 \div q & \text{(h)} \end{array}\right\} \quad (8.7)$$

$$\left.\begin{array}{rll} b \ast x \times y \equiv & (b \ast x) \ast y & \text{(a)} \\ b \ast y \times (b \ast)' d \equiv & d \ast y & \text{(b)} \\ y \times (b \ast)' d \equiv & (b \ast)' d \ast y & \text{(c)} \\ ((d \ast)' q) \times (b \ast)' d \equiv & (b \ast)' q & \text{(d)} \\ (d \ast)' q \equiv & ((b \ast)' q) \div (b \ast)' d & \text{(e)} \\ (d \ast)' b \equiv & 1 \div (b \ast)' d & \text{(f)} \end{array}\right\} \quad (8.8)$$

Equation 8.7 (a) is repeated from Equation 8.1. Equation (b) states that $b \ast -x$ is the reciprocal of $b \ast x$ and is obtained by setting $y \equiv -x$ in Equation (a) and noting that $b \ast x - x \equiv b \ast 0 \equiv 1$. Equation (c) is obtained by applying Equation (a) to $b \ast x + (-y)$ and substituting $1 \div b \ast y$ for $b \ast -y$ as permitted by Equation (b). Equations (d) and (e) are obtained from Equations (a) and (c) respectively by substituting $(b\ast)'\,p$ for x, and $(b\ast)'\,q$ for y, and noting that $b\ast\,(b\ast)'\,z \equiv z$. Equations (f) and (g) are obtained from Equations (d) and (e) respectively by applying the function $(b\ast)'$ to both sides. Equation (h) is obtained by setting $p \equiv 1$ in Equation (g) and noting that $(b\ast)'\,1 \equiv 0$.

Equation 8.8 (a) is repeated from Equation 8.5. Equation (b) is obtained by setting $d \equiv b \ast x$ in Equation (a) and using the fact that x is equal to $(b\ast)'\,d$. Equation (c) is obtained by applying $(b\ast)'$ to both sides of (b), and (d) is obtained by setting $d \ast y \equiv q$ in (c). Equation (e) is obtained from (d) by dividing through by $(b\ast)'\,d$. Equation (f) is obtained by setting $q \equiv b$ in Equation (e) and observing that $(b\ast)'\,b \equiv 1$; Equation (f) states that the base-b logarithm of d is the reciprocal of the base-d logarithm of b.

(Do Exercises 8.6–8.7.)

The Natural Logarithm and Natural Exponential

If b is set equal to $\ast\,1$ in Equation 8.3, then (since $\ast'\ast\,1 \equiv 1$)

$$(\ast\,1) \ast x \equiv \ast\,x$$

In other words, the monadic function $\ast\,x$ is the special case of the base-b exponential where $b \equiv \ast\,1$. Likewise, the function $\ast'\,x$ is equivalent to $(b\ast)'$ where $b \equiv \ast\,1$.

The number

$$\ast\,1 \equiv 1 + 1 + \frac{1}{!\,2} + \frac{1}{!\,3} + \dots$$

is an important constant called *Napier's number* or *the base of the natural logarithms*. It is denoted by e and is approximately equal to 2.71828. The results of the preceding paragraph can now be stated as follows:

$$\left.\begin{aligned} e \ast x &\equiv \ast\,x \\ (e\ast)'\,x &\equiv \ast'\,x \end{aligned}\right\} \qquad (8.9)$$

The function $\ast'\,x$ is called the *natural logarithm*, and the function $\ast\,x$ will be called the *natural exponential*.

Since $\ast$ and $\ast'$ are special cases of $b\ast$ and $(b\ast)'$, their properties can be obtained from Equations 8.7 and 8.8 by omitting all occurrences of any argument preceding the symbol $\ast$. Equations 8.7 and 8.8 therefore yield the following equations:

$$\left.\begin{array}{rlr} \ast x+y \equiv & (\ast x)\times\ast y & \text{(a)} \\ 1 \equiv & (\ast x)\times\ast -x & \text{(b)} \\ \ast x-y \equiv & (\ast x)\div\ast y & \text{(c)} \\ \ast(\ast' p)+\ast' q \equiv & p\times q & \text{(d)} \\ \ast(\ast' p)-\ast' q \equiv & p\div q & \text{(e)} \\ (\ast' p)+\ast' q \equiv & \ast' p\times q & \text{(f)} \\ (\ast' p)-\ast' q \equiv & \ast' p\div q & \text{(g)} \\ -\ast' q \equiv & \ast' 1\div q & \text{(h)} \end{array}\right\} \qquad (8.10)$$

$$\left.\begin{array}{rlr} \ast x\times y \equiv & (\ast x)\ast y & \text{(a)} \\ \ast y\times\ast' d \equiv & d\ast y & \text{(b)} \\ y\times\ast' d \equiv & \ast' d\ast y & \text{(c)} \\ ((d\ast)' q)\times\ast' d \equiv & \ast' q & \text{(d)} \\ (d\ast)' q \equiv & (\ast' q)\div\ast' d & \text{(e)} \\ (d\ast)' e \equiv & 1\div\ast' d & \text{(f)} \end{array}\right\} \qquad (8.11)$$

(Do Exercises 8.8–8.9.)

Tables of Logarithms

Tables of the natural logarithm can be computed by using the polynomial of Equation 7.19. According to Equation 8.11 (e), a table of base-*d* logarithms can be obtained from a table of natural logarithms by dividing the entries by the natural logarithm of *d*. A more efficient method of computing logarithms is developed in Exercise 8.31.

Appendix C gives a table of base-10 logarithms for arguments from 1.00 to 10.00 in steps of .01. More extensive tables of base-10 logarithms are readily available. Tables of the natural logarithms are also available, but logarithms for bases other than 10 and *e* are not usually tabulated.

The base-*b* exponential function $b\ast x$ is not commonly tabulated, since it can be determined from the table for the base-*b* logarithm. This is done by reversing the roles of argument and result, as detailed in Chapter 6, that is, the argument x (or its nearest approximations above and below) is found in the body of the table, and the result is the heading corresponding to that entry. For example, in the table of Appendix C, if $x=3.333$, then $(10\ast)'\,3.33$ is found to be .5224, and $(10\ast)'\,3.34=.5236$. Interpolation finally gives $(10\ast)'\,3.333=0.5228$. Conversely, if $x=0.6667$, the inverse function $10\ast x$ is obtained by locating the bounding entries 0.6665 and 0.6675 in the body of the table and the corresponding headings 4.64 and 4.65. Interpolation then gives $10\ast x=4.642$.

Applications of the Logarithm and Exponential

Multiplication and division. Tables of base-10 logarithms provide an effective basis for performing multiplication and division. Multiplication is based upon Equation 8.7 (d), namely,

$$p \times q \equiv b \ast ((b \ast)' p) + (b \ast)' q$$

For base-10 logarithms this relation becomes

$$p \times q \equiv 10 \ast ((10 \ast)' p) + (10 \ast)' q$$

The product of two positive numbers p and q can therefore be obtained by determining the values of $(10 \ast)' p$ and $(10 \ast)' q$ from a table of base-10 logarithms, adding them to obtain a sum z, and determining from the same table the value of the inverse function $10 \ast z$. This value is the desired product $p \times q$. Similarly, division can be performed by using the following equation (obtained from Equation 8.7 (*e*)):

$$p \div q \equiv (10 \ast ((10 \ast)' p)) - (10 \ast)' q$$

For example, if $p = 5.08$ and $q = 1.89$, then

$$((10 \ast)' p) + (10 \ast)' q = 0.7059 + 0.2765 = .9824$$

and

$$p \times q = 10 \ast .8824 = 9.602$$

Similarly, $p \div q = 10 \ast (0.7059 - 0.2765) = 10 \ast .4294 = 2.688$.

It will be observed that the base-10 logarithms are tabulated only for arguments from 1 to 10. Any positive argument outside this range can be treated as the product

$$x \equiv z \times 10 \ast c$$

where c is an integer so chosen that z is between 1 and 10 and can therefore be found as an argument in the table. Then, since $(10 \ast)'$ $10 \ast c \equiv c$,

$$(10 \ast)' x \equiv ((10 \ast)' z) + c$$

where c is an integer called the *characteristic* of x and $(10 \ast)' z$ is the logarithm of a number z which occurs in the range of the table (Appendix C). The number $(10 \ast)' z$ is called the *mantissa* of x.

For example, if $x = 365$ and $y = .04167$, then

$$\begin{aligned} (10 \ast)' x &= 0.5623 + 2 \\ (10 \ast)' y &= 0.6198 - 2 \\ (10 \ast)' x \times y &= 1.1821 + 0 \end{aligned}$$

and

$$\begin{aligned}(10\ast)'\,x \times y &= 1.1821 + 0\\ &= 0.1821 + 1\end{aligned}$$

Hence

$$\begin{aligned}x \times y &= 1.521 \times 10 \ast 1\\ &= 15.21\end{aligned}$$

Similarly,

$$\begin{aligned}x \div y &= 10 \ast (0.5623 - 0.6198) + (2 - {}^{-}2)\\ &= 10 \ast (0.5623 - 0.6198) + 4\\ &= 10 \ast (1.5623 - 0.6198) + 3\\ &= 10 \ast 0.9425 + 3\\ &= 8.760 \times 10^3\\ &= 8760\end{aligned}$$

(Do Exercises 8.10–8.13.)

Exponentiation. A table of base-10 logarithms can be used to determine the inverse function $10\ast$ and hence can be used to find any power (integral or nonintegral) of the integer 10. It can also be used to determine any power of an arbitrary base d. Setting $b = 10$ in Equation 8.8 (b) yields

$$d \ast y \equiv 10 \ast y \times (10\ast)'\,d$$

In other words, if z is the base-10 logarithm of $d \ast y$, then z is equal to y times the base-10 logarithm of d, and $d \ast y$ is equal to $10 \ast z$.

For example, the ratio r between two successive half tones on the musical scale is such that twelve intervals (one octave) produce the ratio 2:1. Hence $r^{12} = 2$ and $r = 2^{1 \div 12}$. Therefore

$$\begin{aligned}(10\ast)'\,r &= \frac{1}{12} \times (10\ast)'\,2\\ &= \frac{1}{12} \times 0.3010 = 0.0251\end{aligned}$$

(Do Exercises 8.14–8.16.)

Finally, $r = 10 \ast 0.0251 = 1.060$, approximately.

Logarithms of the circular functions. In order to use logarithms to evaluate an expression of the form

$$z \equiv (S\,x) \times C\,x$$

it is necessary first to determine $S\,x$ and $C\,x$ and then to determine their logarithms. Thus

$$z \equiv 10 \ast ((10\ast)'\,S\,x) + (10\ast)'\,C\,x$$

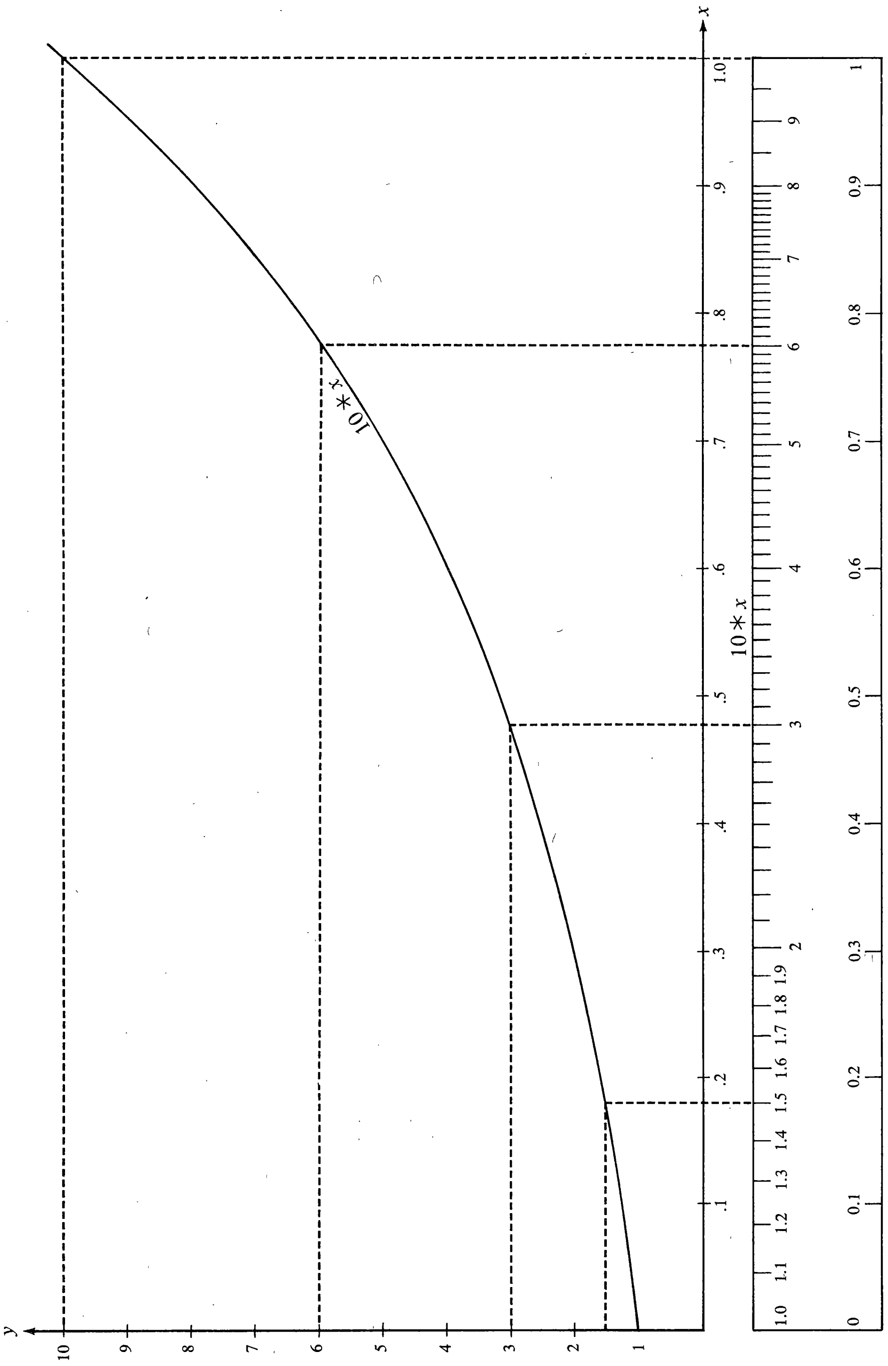


Figure 8.2 The rule representation of the function 10 * *x*

To eliminate one of these steps, tables of the composite functions $(10*)'S$ and $(10*)'C$ are often provided. The functions are called *log sine*, *log cosine*, *log tangent*, and so forth.

(Do Exercise 8.17.)

The slide rule. Any function Fx can be represented by recording corresponding values of argument and function along a straight rule, as shown in Figure 8.2 for the function $10*x$. For example, the value of $10*0.8$ can be found by locating 0.8 on the lower scale and reading off the (approximate) corresponding value 6.3 on the upper scale.

The inverse function $(10*)'y$ can be determined from the same rule by locating the argument y on the upper scale. Thus for $y=3$, the value of the base-10 logarithm $(10*)'y$ is approximately 0.475.

The rule of Figure 8.2 can therefore be used for multiplication. For example, the product 1.6×2.5 can be obtained as follows:

$$(10*)'\,1.6\equiv0.2$$
$$(10*)'\,2.5\equiv0.4$$

Therefore,

$$(10*)'\,1.6\times2.5\equiv((10*)'\,1.6)+((10*)'\,2.5)\equiv0.2+0.4\equiv0.6$$

Finally,

$$1.6\times2.5\equiv10*0.6\equiv4.0$$

as read from the rule at the point $x=0.6$.

Since the scale for x (that is, for the logarithm) is uniform, the addition of the logarithms can be performed directly on the rule itself by using a draftsman's divider. Thus, if a divider is set to span a length from the beginning of the rule to the point 1.6 on the upper scale and is then moved to place the first leg at the point 2.5 on the upper scale, the second leg will rest at the value 0.6 on the lower scale and at the value 4.0 on the upper scale, that is, at the value of the sum of the logarithms on the lower scale and at the value of the product on the upper scale.

Adding distances determined by the upper scale and reading the result on the upper scale therefore determines a product directly. Gunter, who first devised this method of multiplication in 1620, used dividers to perform the addition of logarithms as outlined above. About ten years later Wingate introduced the use of two similar rules laid side by side to perform the multiplication as illustrated in Figure 8.3 for the same arguments 1.6 and 2.5.

Somewhat later an instrument maker fastened two rules and a cursor together in a sliding arrangement similar to the modern slide rule shown in Figure 8.4. The modern slide rule is accurate to about three decimal places and provides a number of additional scales representing various useful functions, such as the logarithms of the circular functions.†

The slide rule, like a table of logarithms, represents the function $(10 \ast)' x$ only for a limited range of argument values, namely, for $1 \leq x \leq 10$. The usable range of the slide rule is extended exactly as for the log table: an integral characteristic represents a factor of the form $10 \ast c$. If a series of products moves the result off the upper end of the scale of a slide rule, the cursor is moved back the length of one rule. This action is compensated for by making a final multiplication by ten.

(Do Exercises 8.18–8.22.)

The Family of Exponential Functions

By considering complex arguments, it is possible to express all the circular functions and also the hyperbolic functions A and B (defined in Equations 5.11 and 5.12) in terms of the single function $\ast\, x$. This brings out more clearly the relations between the functions studied thus far. It also gives further evidence of the importance of complex numbers.

It will first be necessary to review briefly the elementary properties of complex numbers. They are usually first encountered in mathematics as an extension of the real numbers that is required to ensure that every polynomial will have a root.

The need for such an extension arises in the case of the quadratic equation

$$x^2 + (2 \times b \times x) + c = 0$$

which has the general solution

$$x \equiv (-b) \pm \sqrt{} b^2 - c$$

If $(b^2 - c) < 0$, there is no real number r such that $r = \sqrt{} b^2 - c$. However, in that case x is equal to

$$(-b) \pm (\sqrt{} c - b^2) \times \sqrt{} -1$$

---

†For an interesting account of the development of the slide rule, see F. Cajori, "A History of the Logarithmic Slide Rule," in W. W. R. Ball *et al*, *String Figures and Other Monographs* (Chelsea, 1960).

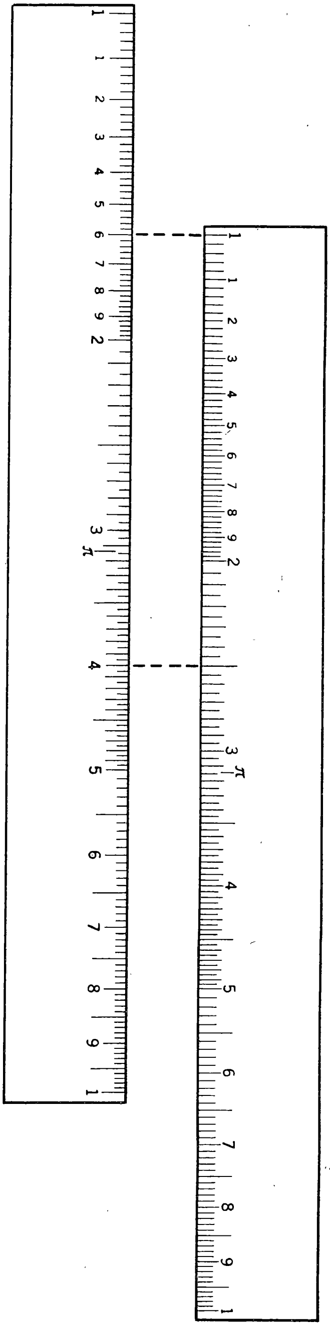


Figure 8.3 Multiplication by means of rules

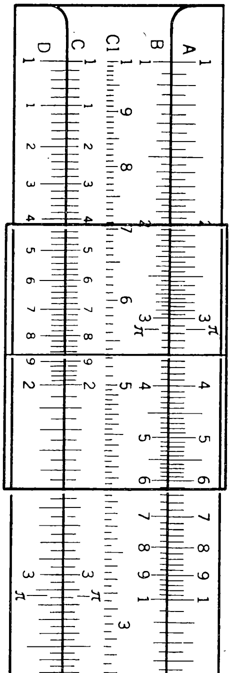


Figure 8.4 A modern slide rule

and by introducing the symbol i for $\surd^{-}1$, the solution for x can be written as

$$x \equiv (-b) \pm (\surd c - b^2) \times i$$

Any number of the form $f + g \times i$ is called a *complex number*, the real number f is called its *real* part, and the real number g is called its *imaginary* part. Complex numbers obey the normal rules of arithmetic, with the added characteristic that

$$i \times i \equiv (\surd^{-}1)^2 \equiv {}^{-}1$$

Thus, if $p \equiv a + b \times i$ and $q \equiv c + d \times i$, then

$$p + q \equiv (a + c) + (b + d) \times i$$

and

$$\begin{aligned} p \times q &\equiv (a \times c) + (a \times d \times i) + (b \times c \times i) + (b \times d \times i \times i) \\ &\equiv ((a \times c) - (b \times d)) + ((a \times d) + (b \times c)) \times i \end{aligned}$$

(Do Exercises 8.23 – 8.24.)

Since the polynomial $\boldsymbol{c}\,\Pi\, x$ is evaluated solely by multiplication and addition, and since both these operations are defined for complex arguments, it is meaningful to consider the polynomial $\boldsymbol{c}\,\Pi\, p$ with a complex argument p, even for a polynomial of unlimited degree.

In particular, the exponential

$$* x \equiv 1 + x + \frac{x^2}{!2} + \frac{x^3}{!3} + \frac{x^4}{!4} + \ldots$$

for the imaginary argument $x \times i$ becomes

$$\begin{aligned} * x \times i &\equiv 1 + (x \times i) + \frac{(x \times i)^2}{!2} + \frac{(x \times i)^3}{!3} + \ldots \\ &\equiv \left(\left(1 - \frac{x^2}{!2}\right) + \left(\frac{x^4}{!4} - \frac{x^6}{!6}\right) + \ldots\right) \\ &\qquad + i \times \left(\left(x - \frac{x^3}{!3}\right) + \left(\frac{x^5}{!5} - \frac{x^7}{!7}\right) + \ldots\right) \\ &\equiv (C\,x) + i \times S\,x \end{aligned}$$

where C and S are the cosine and sine functions. Hence

$$*\,x\times i\equiv (C\,x)+i\times S\,x \tag{8.12}$$

$$*-x\times i\equiv (C\,x)-i\times S\,x \tag{8.13}$$

$$C\,x\equiv\frac{1}{2}\times(*\,x\times i)+*-x\times i \tag{8.14}$$

$$S\,x\equiv\frac{1}{2\times i}\times(*\,x\times i)-*-x\times i \tag{8.15}$$

Thus the sine and cosine functions can be expressed in terms of the exponential function in a manner analogous to that shown for the hyperbolic functions *A* and *B* in Chapter 5:

$$A\,x\equiv\frac{1}{2}\times(*\,x)+*-x \tag{5.11}$$

$$B\,x\equiv\frac{1}{2}\times(*\,x)-*-x \tag{5.12}$$

Moreover, since

$$A\,x\equiv 1+\frac{x^2}{!2}+\frac{x^4}{!4}+\dots$$

then

$$\begin{aligned}A\,x\times i&\equiv\left(1-\frac{x^2}{!2}\right)+\left(\frac{x^4}{!4}-\frac{x^6}{!6}\right)+\dots\\&\equiv C\,x\end{aligned}$$

This and similar arguments lead to the following set of relations:

$$\left.\begin{aligned}A\,x\times i&\equiv C\,x\\B\,x\times i&\equiv i\times S\,x\\C\,x\times i&\equiv A\,x\\S\,x\times i&\equiv i\times B\,x\end{aligned}\right\} \tag{8.16}$$

The foregoing relations, together with the addition theorems for the sine and cosine, can be used to derive analogous addition theorems for the hyperbolic functions *A* and *B*.

Since $*\,x\times i\equiv(C\,x)+i\times S\,x$, it follows that

$$*\,\pi\times i\equiv(C\,\pi)+i\times S\,\pi$$

But $C\,\pi={}^{-}1$ and $S\,\pi=0$. Hence $e*\pi\times i\equiv{}^{-}1$, or

$$1+e*\pi\times i\equiv 0$$

(Do Exercises 8.25–8.34.)

Exercises

8.1 (a) Use Equation 7.19 to compute $*'2$ to four decimal places.

(b) Use Equations 8.3 and 5.8 and the result of part (a) to compute $2*x$ for $x = {}^{-}1, 0, \frac{1}{2}, 1, 2,$ and $\frac{2}{3}$.

(c) Verify the results of part (b) by comparing with the value of 2 raised to the appropriate powers.

8.2 (a) Write a program to compute the natural logarithm $*'x$ to within a tolerance t.

(b) Execute your program for $x = 2$ and $t = .0001$.

8.3 Evaluate the left and right sides of Equation 8.5 for the case $b = 2$, $x = .3$, and $y = .4$, and compare the results.

8.4 Use Equation 8.6 to compute the following to three decimal places:

(a) $(2*)'3$ (*b*) $(3*)'2$ (c) $(2*)'2$

8.5 Extend the program of Exercise 8.2 to compute the base-*b* logarithm to within a specified tolerance *t*.

8.6 Choose suitable values for the arguments in Equations 8.7 and 8.8, and use them to test the equations by computing and comparing the right and left sides of each equation.

8.7 State in words the relations expressed by each of the equations in 8.7 and 8.8; for example, Equation 8.7 (f) states that the base-*b* logarithm of a product is the sum of the base-*b* logarithms of the factors.

8.8 Repeat Exercise 8.6 for Equations 8.10 and 8.11.

8.9 Repeat Exercise 8.7 for Equations 8.10 and 8.11.

8.10 Use the log table of Appendix C to compute the following products:

(a) 2×3
(b) 3×3
(c) 3×10
(d) 3.14×27.24
(e) $2138\times .00124$

8.11 Extend the result of Equation 8.7 (d) to three factors; that is, prove that

$$p\times q\times r \equiv b*((b*)'p)+((b*)'q)+(b*)'r$$

(b) Use log tables to compute $3.14\times 2.718\times 365$.

(c) Use log tables to compute $365\times 24\times 60\times 60$.

8.12 Use log tables to compute the following:

(a) $6\div 3$
(b) $24\div 6$
(c) $3\div 6$
(d) $6\div 24$
(e) $2.718\div 3.14$

8.13 Use log tables to compute the following:

(a) $\frac{32\times 2.7\times 2.7}{2\times 11.6}$

(b) $(32\div 11.6)\times(17\div 16)\times(9\div 11)$

8.14 Use log tables to compute the following:

(a) $16 * 4$

(b) $16 * \frac{1}{4}$

(c) $3.14 * 2.718$

(d) $(3.14 * 2.718)\times 2.7 * 1.6$

(e) $(1.3 * 2.2) * 5$

(f) $1.3 * 2.2\times 5$

(g) $2 * \frac{1}{12}$

8.15 Use log tables to evaluate the following expressions and compare the results:

(a) $3.14 * 3$ (b) $3.14\times 3.14\times 3.14$

8.16 Account for the convenience of tables of base-10 logarithms as compared with tables of logarithms using other bases.

8.17 Use tables of log sine, log cosine, and log tangent to evaluate the following expressions.

(a) $\left(S\,45\times\frac{\pi}{180}\right)\times C\left(45\times\frac{\pi}{180}\right)$

(b) $\left(S\,30\times\frac{\pi}{180}\right)\div C\left(30\times\frac{\pi}{180}\right)$

(c) $\left(T\,30\times\frac{\pi}{180}\right) * 2$

(d) $(S\,1.3)\times(C\,1.3)\div T\,0.6$

8.18 Repeat Exercise 8.10 using a slide rule instead of log tables.

8.19 Repeat Exercise 8.12 using a slide rule.

8.20 Repeat Exercise 8.14 using a slide rule.

8.21 Use a slide rule to evaluate the following:

(a) $\frac{12.681\times 64\times 132}{6\times 15.40\times 27}$

(b) $\frac{(3.14)^2\times\surd\,17\times 14.3}{2.718\times 14 * 1\div 3}$

8.22 Repeat Exercise 8.17 using a slide rule.

8.23 Compute the sum and the product of each of the following pairs of complex numbers:

(a) $(3+4\times i)$ and $(6+10\times i)$

(b) $(3-5\times i)$ and $(3+5\times i)$

(c) $(0+5\times i)$ and $(0+6\times i)$

(d) $(3+0\times i)$ and $(5+0\times i)$

8.24 Write programs to determine x and y such that

(a) $x+y\times i\equiv(a+b\times i)+(c+d\times i)$

(b) $x+y\times i\equiv(a+b\times i)\times(c+d\times i)$

8.25 Evaluate the following:

(a) $3+2\times(4+2\times i)+2\times(4+2\times i)^2$

(b) $(3\ ,2\ ,2)\ \Pi\ (4+2\times i)$

(c) $\boldsymbol{c}\,\Pi\,(4-2\times i)$ for $\boldsymbol{c}\equiv 3\ ,{}^-2\ ,2\ ,{}^-1$

(d) $\boldsymbol{c}\,\Pi\,2\times i$ for $\boldsymbol{c}\equiv 3\ ,{}^-2\ ,2\ ,{}^-1$

8.26 Write a program to determine $\boldsymbol{x}$ such that

$$(\boldsymbol{x}_1+\boldsymbol{x}_2\times i)\equiv\boldsymbol{c}\ \Pi\ (\boldsymbol{a}_1+\boldsymbol{a}_2\times i)$$

8.27 For each of the following values of x, compute the value of $\ast\ (x\times i)$ to three decimal places, and compare the two parts of the result with $S\,x$ and $C\,x$ as found in the table in Appendix B.

(a) $x=\frac{\pi}{6}$ (b) $x=0.2$ (c) $x={}^-0.2$

8.28 (a) Since $S\,x\equiv\frac{(\ast\,x\times i)-\ast\,-x\times i}{2\times i}$, an addition theorem for the sine function can be obtained from the relation

$$S\,x+y\equiv\frac{(\ast\ (x\times i)+y\times i)-\ast\ (-\ (x\times i)+y\times i)}{2\times i}$$ by

using the addition theorem for the exponential to evaluate the expressions $(\ast\ (x\times i)+y\times i)$ and $\ast\ (-\ (x\times i)+y\times i)$. Show that the result agrees with the addition theorem already derived for the sine.

(b) Perform a similar derivation and check for the addition theorem for the cosine.

8.29 Use the method of Exercise 8.28 to check the following addition theorems for the hyperbolic functions:

(a) $A\,x+y\equiv((A\,x)\times A\,y)+(B\,x)\times B\,y$

(b) $B\,x+y\equiv((A\,x)\times B\,y)+(B\,x)\times A\,y$

8.30 Derive an addition theorem for the hyperbolic tangent U as defined in Exercise 7.28.

8.31 The terms of the polynomial expression for $\ast'\,n$ (Equation 7.19) decrease rather slowly for large values of the argument. However, the identity

$$\ast'\,n\equiv(\ast'\,q)+\ast'\,n\div q$$

(obtained from Equation 8.10 (g)) permits the calculation of the natural logarithm of n as the sum of two natural logarithms each computed for an argument smaller than n.

A table of natural logarithms for arguments from 1 to l in steps of g can be computed in this way by using the last entry computed for computing the next.

(a) Write a program using the foregoing scheme to make a table of natural logarithms for arguments from 1 to l in steps of g.

(b) Modify the program of part (a) to make a table of base-10 logarithms for the same range of arguments.

8.32 A cylindrical water tank with a vertical axis is fitted with a discharge vent at the bottom. The rate of discharge (in gallons per minute) at any instant is proportional to the pressure at the bottom and hence to the number of gallons remaining in the tank. If this constant of proportionality is p and if the amount of water remaining in the tank at time t is $W\ t$, then

$$(D\ W)\ t \equiv -p \times W\ t$$

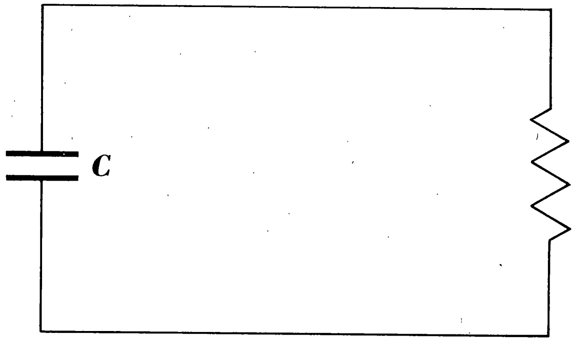
Let f be the number of gallons present in the tank when the vent is first opened at time $t = 0$.

(a) Derive an expression for $W\ t$ as a function of f, p, and t.

(b) Determine the number of gallons remaining in the tank after ten minutes if $f = 500$ gallons and $p = 0.1$.

(c) Determine (as a function of p) the "half-life" of the tank's contents, that is, the time at which exactly one-half of the original contents remains.

8.33 In the electrical circuit shown in the diagram, the rate of discharge at any instant is equal to $\frac{V\ t}{r}$ coulombs per second, where r is the resistance in ohms and $V\ t$ is the



voltage at time t. The voltage is in turn equal to $\frac{Q\ t}{c}$, where c is the capacity (in farads) of the condenser and $Q\ t$ is the charge (in coulombs) on the condenser at time t. Therefore

$$(D\ Q)\ t \equiv -\frac{V\ t}{r} \equiv -\frac{Q\ t}{c \times r}$$

The initial charge at time $t = 0$ is f coulombs.

(a) Determine the charge as a function of t, f, c, and r.

(b) Determine the time required to reduce the charge to one-half of its initial value.

(c) The rate of dissipation of energy in the resistor at any instant is equal to the product of the voltage and current, and hence to $\frac{(V\ t)^2}{r}$. The total energy dissipated is therefore equal to the total area under the graph of the function $\frac{(V\ t)^2}{r}$. Obtain an expression for the total energy dissipated and observe that it does not depend on the value of r.

8.34 The maximum value of the function $H\ n \equiv n \ast 1 \div n$ investigated in Exercise 3.34 can be determined by deriving the slope of the function and then finding the point at which the slope is zero. However, since the base n is not a constant, Equation 8.2 cannot be applied to obtain the slope; and since the exponent $1 \div n$ is not a constant, Equation 5.4 cannot be applied.

The slope of the function $H\ n$ can be derived by first considering the natural logarithm of the function.

(a) Show that $(\ast'\ H)\ n \equiv (1 \div n) \times \ast'\ n$.

(b) Show that $(D \ast'\ F)\ n \equiv \frac{(D\ F)\ n}{F\ n}$ for any function F. (Use Equations 7.9 and 7.15.)

(c) Show that $\frac{(D\ H)\ n}{H\ n} \equiv \left(\frac{1}{n} \times \frac{1}{n}\right) + (\ast'\ n) \times \frac{{}^{-}1}{n^2}$. (Use part (a) and Equations 5.6, 7.4, and 7.15.)

(d) Show that $(D\ H)\ n \equiv (n \ast 1 \div n) \times \frac{1}{n^2} \times 1 - \ast'\ n$.

(e) Show that the slope of the function $n \ast 1 \div n$ is zero for $n = e$.

(f) Determine the maximum value of the function $n \ast 1 \div n$.

Chapter Nine

Automatic Program Execution

Manual execution of a program is frequently tedious and time consuming, and it is therefore convenient to employ the modern automatic computer, which executes programs rapidly and automatically. The computer is an interesting device whose operation can itself be described and studied by means of programs. In the study of elementary functions, however, the computer is of interest only as a tool for the execution of programs.†

The Typewriter

The computer is controlled by a typewriter having the characters shown in Figure 9.1. A statement entered on the typewriter is executed by the computer, and results can be printed by the same typewriter.

Since the typewriter provides only one set of letters, it is impossible to distinguish scalars, vectors, and matrices by type of letter (lightface, boldface, and boldface capital) as is done in the text. Different types of letters are convenient for reading but are not essential, since the distinctions are implicit in any statement; for example, in the statement

$$X \leftarrow 3\,,2\,,5$$

X is necessarily a vector. It is, however, necessary to choose different symbols for various scalars, vectors, and functions occurring in a pro-

---

†The computer system described here is an IBM 7090 computer provided with an IBM 1050 typewriter terminal and an interpreter program written by L. M. Breed and P. S. Abrams. Similar systems employing other computers and other interpreter programs may differ in detail; the manual for any particular system should therefore be consulted.

```
¨¯<≤=≥>≠∨∧-÷
1234567890+×

?ωερ~↑↓ι○*→
QWERTYUIOP←

α⌈⌊_∇∆∘'⎕()
ASDFGHJKL[]

⊂⊃∩∪⊥⊤|;:\
ZXCVBNM,./
```

Figure 9.1 Character set of IBM Selectric Typing Element #1167988

gram. A string of alphabetic characters with no intervening spaces or nonalphabetic characters is treated as a single symbol for a variable or function. An alphabetic symbol for a variable or function must be separated by at least one space (on each side) from digits and from other alphabetic characters.

Since typing proceeds on a single line, an index for a vector cannot be typed as a subscript but, instead, must be indicated by enclosing it in brackets. Thus $\boldsymbol{x}_i$ is typed as $X[I]$. All such symbol substitutions are shown in Appendix D. Certain symbols are produced by backspacing and overstriking. For example, ⌽ is formed by striking ○, backspace, and | in succession.

When a statement has been typed, the carriage return initiates its immediate execution. For example, typing of the sequence

$$
\begin{aligned}
&X \leftarrow 7\\
&Y \leftarrow 3\\
&Z \leftarrow (X-Y)\times X+Y
\end{aligned}
$$

will assign the values 7, 3, and 40 to the variables X, Y, and Z respectively. However, since these variables are represented only in the computer's memory, their values cannot be observed directly.

The value of any variable can be displayed by executing a statement that causes it to be printed by the typewriter. The *quad* symbol ⎕ is assigned to the typewriter and then treated as a variable; thus typing of the statement

$$\square \leftarrow Z$$

will cause the number 40 (that is, the value of Z) to be automatically typed out. Similarly, typing of the statement

$$\square \leftarrow 572 \times 1319$$

would be followed immediately by the automatic typing of the number

$$754468$$

Results produced by the computer are accurate to some fixed number of significant digits (typically eight or more) depending upon the particular computer system in use.

(Do Exercise 9.1.)

Branches

In typing programs for the computer, an unconditional branch to the statement whose number is the value of X is denoted by

$$\rightarrow X$$

A conditional branch is denoted by

$$\rightarrow U / X$$

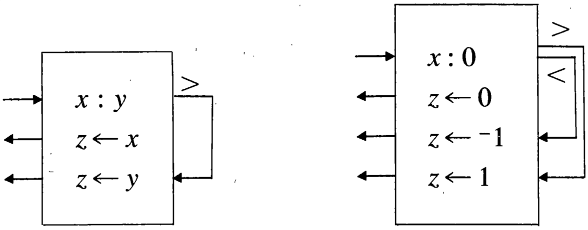
where U is a logical vector and X is a vector of statement numbers. The compression selects from X the components indicated by the nonzero components of U. If exactly one component is selected, it determines the statement executed next; if none are selected, the normal successor is executed next; and if two or more are selected, the branch is invalid. For example, step 1 of Program 9.2 (a) (reproduced from Exercise 2.9 (a)) would be typed as

$$\rightarrow (X > Y) / 3$$

and step 1 of Program 9.2 (b) would be typed as

$$\rightarrow ((X < 0) , X > 0) / 3 , 4$$

A branch to any statement number outside the range of statement numbers of the program signifies completion of the program. This includes the case of executing the last statement of a program which (provided this statement is not itself a branch) is, in effect, succeeded by a statement whose index is outside the permitted range.



(a) (b)

Program 9.2

Definition of Functions

Any new function can be defined in the manner indicated in Appendix D and detailed in Chapter 3. The symbol ∇ (an inverted Greek delta, called *del*) is typed at the beginning of the line on which the new function is named. The statements of the program defining the function follow on successive lines, and are followed (on a separate line) by the symbol ∇, which terminates the definition.

For example, typing of the sequence

```
     ∇G←M U N
[1]  G←M
[2]  M←M|N
[3]  N←G
[4]  →(M≠0) / 1
[5]  ∇
```

defines the function U such that $M\ U\ N$ is the greatest common divisor of M and N (see Program 3.6). The numbers in brackets are statement numbers, which are typed automatically by the computer. Subsequent typing of

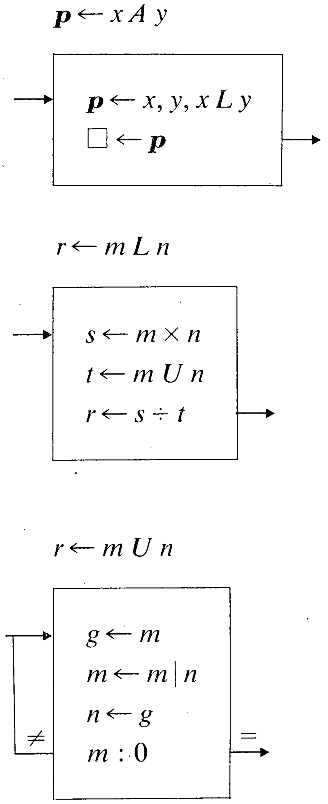
```
□←42 U 30
```

will evaluate U for the arguments 42 and 30 and type out the result 6, that is, the greatest common divisor of 42 and 30.

Once a function has been defined, it behaves like the basic functions listed in Appendix D. In particular, it can be used in the definition of further functions. For example, Figure 9.3 (a) shows the definition of the greatest common divisor function U, the least common multiple function L (using U in its definition), and a function A (using both L and U). Figure 9.3 (b) shows what must be typed to define these same functions. Typing the lines of Figure 9.3 (b) and then the line

```
B←21 A 28
```

will evaluate the function A for the arguments 21 and 28, specifying
(Do Exercise 9.2.) B as 21, 28, 84, and then typing out the value of B.



(a)

```
      ∇ P←X A Y
[1]     P←X, Y, X L Y
[2]     □←P
[3]     ∇

      ∇ R←M L N
[1]     S←M × N
[2]     T←M U N
[3]     R←S ÷ T
[4]     ∇

      ∇ R←M U N
[1]     G←M
[2]     M←M|N
[3]     N←G
[4]     → (M ≠ 0) / 1
[5]     ∇
```

(b)

Values of △ for 21 A 28
1
1 , 1
1 , 2
1 , 2 , 1
1 , 2 , 2
1 , 2 , 3
1 , 2 , 4
1 , 2 , 1
1 , 2 , 2
1 , 2 , 3
1 , 2 , 4
1 , 3
2

(c)

Figure 9.3 Function definition

Correction and Display of Programs

An error in a statement being typed can be corrected *before* pressing the carriage return by backspacing to the point of correction, pressing the *linefeed* button (which advances the paper), and retyping on the new line everything from the point of correction on. In order to mark the point of correction (which otherwise would not be recorded on the paper if the correction began with spaces) the linefeed is followed by the automatic typing of the symbol ∨ (an inverted caret), a backspace to reposition the carriage, and a further automatic linefeed. For example, the printed record

```
A←A X B+C
     ∨
     ×B+C
```

would result from typing the first line shown, backspacing to the *X*, pressing the linefeed, and then typing the correction shown on the bottom line.

In short, the linefeed types the symbol ∨ and "erases" everything from that point on. Further corrections can be made in the same manner. It must be emphasized that backspacing alone does not erase.

In defining a function, the number of each statement is typed at the left in brackets. These statement numbers are typed automatically by the computer, but to make corrections convenient they can be overridden by typing an alternative statement number in brackets before typing the statement. A statement number can be any number with at most two digits to the right of the decimal point and at most two digits to the left of it. (Fractional line numbers permit the insertion of statements between existing statements.) The statement numbers automatically typed are increased by 1 or, if the number is fractional, by 1 in the least significant digit position.

When the definition of the function is concluded (by typing ∇), the following actions occur in the indicated order:

(1) if any statement number is repeated, the earlier occurrences are deleted and only the last associated statement is retained;
(2) if any statement is empty (that is, the bracketed statement number was followed only by a linefeed and a carriage return, it is deleted;
(3) the statements are reordered according to their statement numbers, and the statement numbers are replaced by the integers 1, 2, 3, and so on.

Thus it is easy to make replacements, deletions, and insertions of statements during the definition of a function. For example, the typing shown on the left of Figure 9.4 will produce the function shown on the right. The line headed by [5] replaces statement 2, the next line inserts the statement $\square \leftarrow P$ between statements 2 and 3, and the next line deletes statement 4.

After the definition of a function has been concluded (by typing the final ∇), corrections can be made by returning to the function definition mode, that is, by typing ∇ and the name of the function to be corrected. This will be followed by the automatic typing of the

```
        ∇ BINOMIAL N
[1]     P←1
[2]     P←P,0+0,P
[3]     →(N>P[2])/2
[4]     □←P
[5]     [2]P←(P,0)+0,P
[3]     [2.1]□←P
[3.1]   [4]
             ∨
[5]     ∇
```

```
        ∇ BINOMIAL N
[1]     P←1
[2]     P←(P,0)+0,P
[3]     □←P
[4]     →(N>P[2])/2
[5]     ∇
```

Figure 9.4 Corrections in function definition

statement number one greater than the number of statements in the definition of the function. Correction then proceeds in the usual manner.

For example, if the function *U* of Figure 9.3 has been defined, then typing ∇U will cause the following automatic type-out of [5]. An entire function can be deleted by deleting every statement in its definition.

In the function definition mode, the typing of [*n* □] will cause the automatic typing of statement *n* of the function definition; and the typing of [□] will cause the typing of the entire function definition, including the name line. Typing can be stopped by pressing the *attention* button, which returns the system to the function definition mode.

(Do Exercises 9.3 to 9.5.)

Interrupted Execution

In executing a program either manually or on a computer, it is necessary to keep track of what statement is currently being executed. In the computer a special variable called an *instruction counter* serves this purpose. It is denoted by the Greek letter Δ (delta), and its value is the number of the statement being executed. The value of Δ is increased by one at the completion of each statement, except that it may be respecified by the value of the expression occurring in a branch statement.

In the example of Figure 9.3, functions are used on several *levels*; thus the main or *highest* level function *A* uses the function *L*, which in turn uses the function *U*. The instruction counter Δ must keep track of the current instruction at each level; it is therefore a vector of varying dimension, with one component for each level and with the highest level counter first. For example, in executing A for the arguments 21 and 28, the instruction counter takes on the sequence of values shown in Figure 9.3 (c).

The execution of a function may be interrupted because of an erroneous statement in the function or because the operator presses the attention button. If the attention button is pushed, execution will stop as soon as the end of a statement is reached. The name of the lowest level function being executed will then be typed out, followed by the number (enclosed in brackets) of the next statement to be executed.

Once a program has been interrupted for any reason, statements can be executed from the keyboard. For example, typing

$$\square \leftarrow X$$

will cause the current value of the variable X to be displayed. Typing

$$\square \leftarrow \Delta$$

will cause the display of the current value of the instruction counter vector. Execution of the interrupted program can be resumed† by typing a branch $\rightarrow R$. Execution will then resume with statement R of the lowest level program. Alternatively, the instruction counter can be "reset" by the statement

$$\Delta \leftarrow \iota\, 0$$

The type-out of a vector quantity (by a statement of the form $\square \leftarrow X$, where X is a vector) can be interrupted at the completion of any component by pushing the attention button. Typing will simply stop; statements can then be executed from the keyboard, and the execution can be resumed in the usual manner. However, typing a space will cause the typing to continue to completion of the type-out statement; at that point an end-of-statement interruption will occur and will be indicated in the usual way by an automatic type-out of the function name and statement number.

Invalid Statements

If a statement that is being executed is not meaningful, its execution will be interrupted, and the computer will then type

---

†In some systems defining or correcting a function while execution is interrupted also sets Δ to the value $\iota\, 0$ and execution of the interrupted program cannot be resumed.

(1) one of the messages in the left-hand column of Table 9.5 to indicate the nature of the fault,
(2) the invalid statement, preceded (if applicable) by the name of the function in which it occurs and its statement number in brackets, and
(3) a caret directly below the symbol at which the fault was detected

For example, if the statement $Z \leftarrow 3 + \times 9 - 5$ occurred as statement 3 in the definition of a function F, then an attempt to execute F would result in an interruption of the execution and the typing of

```
SYNTAX ERROR
F [3] Z←X+×9
          ∧
```

indicating that the evaluation of the expression stopped at the point where the first argument of the multiplication was found to be missing. **(Do Exercise 9.6.)**

Type of error	Significance
BRANCH	Result of branch expression is of dimension other than 0 or 1
FUNCTION	Improper expression in function definition, correction, or display
CHARACTER	Illegal character in input
INDEX	Value of subscript expression not in the range of indices of the variable indexed
LABEL	Improper use of colon (which delimits a label)
LENGTH	Dimensions of arguments do not match
M FULL	Computer memory full
NAME	Allowable length of the name of a function or variable is exceeded
RANGE	Argument value out of range of function (for example, division by 0 or a nonintegral value for index or branch)
RANK	Rank of an argument X (that is, $\rho\,\rho\,X$) is too large for the function; or nonmatching ranks, for example, $\boldsymbol{A} + \boldsymbol{b}$
SYNTAX	Ill-formed statement
VALUE	Value of variable has not been specified

Table 9.5 Error messages

Statement Labels

Modification of a program produced by inserting or deleting statements changes the statement numbers of subsequent statements. It may therefore require changes in the branch statements occurring in the program.

If each statement number occurring in branch statements is replaced by a corresponding variable which is separately specified to give it the appropriate numerical value, then any change in statement numbering can be accommodated (without changing the branch statements themselves) by simply changing the values of these statement number variables. Moreover, the value of any variable used in this manner can be automatically defined if the variable is associated with the statement by typing the variable and a colon to the left of the statement (as shown in Figure 9.6); the variable is called the *label* of the statement. If insertions or deletions in the program change any statement numbers, the value assigned to the associated label is automatically changed at the conclusion of the definition.

```
    ∇ R←M U N
[1]   F:G←M
[2]   M←M|N
[3]   N←G
[4]   →(M≠0)/F
[5]   ∇
```

Figure 9.6 Statement labels

(Do Exercise 9.7.)

Literals

The ability to type out a message such as *FINISHED* or *X* is sometimes needed. Since the execution of the statement

```
□←X
```

types the *value* of the *variable X*, it is necessary to indicate explicitly if the *literal symbol X* is to be typed instead. A literal symbol is indicated as in ordinary English, by enclosing the symbol in quotation marks. Thus the execution of

```
□←'X'
```

types out

$$X$$

and the execution of

$$\square \leftarrow \text{'}FINISHED\text{'}$$

types out

(Do Exercise 9.8.)

$$FINISHED$$

Analysis of a Program

Any newly defined function must be analyzed carefully, and perhaps modified, to ensure that it will produce the intended results. The tool provided for this analysis is the *trace.*

A trace is an automatic type-out of information generated by the execution of a program as it progresses. In a complete trace of a function F, the number of each statement executed is typed out enclosed in brackets, preceded by the symbol F, and followed by the value assigned to the result variable of the statement. For example, Figure 9.7 shows a complete trace for the case $B \leftarrow 21\ U\ 28$, where U is the function defined in Figure 9.3.

The tracing of a function F is controlled by the *trace vector* for F, denoted by ΔF. Statement k of F is traced if and only if some component of ΔF is equal to k. Thus if ΔF is specified by executing the statement

$$\Delta F \leftarrow 2\ ,3\ ,5$$

then statements 2, 3, and 5 will be traced in any subsequent execution of F. Tracing of F can be discontinued by executing the statement $\Delta F \leftarrow \iota\, 0$.

A specification of a trace control vector can be useful within a program as well as in direct execution from the typewriter. For example, the statement

$$\Delta Q \leftarrow (J > 10) \times 3\ ,5\ ,12$$

incorporated in a program Q would cause the tracing of statements 3, 5, and 12 of program Q to be instituted only after the statement is executed with the value of the variable J exceeding ten.

(Do Exercises 9.9 and 9.10.)

```
U [1]  21
U [2]  7
U [3]  21
U [4]  1
U [1]  7
U [2]  0
U [3]  7
U [4]
```

Figure 9.7 Complete trace of 21 *U* 28

Other Basic Functions

The computer system provides a number of useful functions that were not discussed in previous chapters because they were not essential to the development. Their treatment here will be brief; the reader can clarify doubtful points by experimenting with the functions on the computer†.

The argument *i* in the expression $\boldsymbol{x}_i$ can be a vector as well as a scalar. For example, if $\boldsymbol{x} \equiv 6 , 8 , 10 , 12 , 14$ and $\boldsymbol{i} \equiv 3 , 1 , 4$, then $\boldsymbol{x}_{\boldsymbol{i}} \equiv 10 , 6 , 12$.

The dyadic function ↑ is called *left rotation* and is defined as follows:

$$k \uparrow \boldsymbol{x} \equiv \boldsymbol{x}_{1+(\rho \boldsymbol{x}) | -1 + k + \iota \rho \boldsymbol{x}}$$

Right rotation is denoted by ↓ and is defined as follows:

$$k \downarrow \boldsymbol{x} \equiv \boldsymbol{x}_{1+(\rho \boldsymbol{x}) | (-1 - k) + \iota \rho \boldsymbol{x}}$$

For example, $2 \uparrow 1 , 2 , 3 , 4 , 5 \equiv 3 , 4 , 5 , 1 , 2$ and $2 \downarrow 1 , 2 , 3 , 4 , 5 \equiv 4 , 5 , 1 , 2 , 3$.

The *prefix vector* $n \, \alpha \, j$ is a logical vector of dimension *n* whose first *j* components are equal to 1. More precisely, $n \, \alpha \, j \equiv j \geq \iota \, n$. Similarly, the *suffix vector* $n \, \omega \, j$ is defined as $\text{⌽} \, j \geq \iota \, n$. For example, $5 \, \alpha \, 3 \equiv 1 , 1 , 1 , 0 , 0$, and $5 \, \omega \, 3 \equiv 0 , 0 , 1 , 1 , 1$.

The monadic function ~ (*logical negation*) and the dyadic functions ∨ and ∧ (*or* and *and*) are defined only for logical values of their arguments (that is 0 and 1). They are defined as follows:

---

†Further discussion of these functions and their application to computers and data processing can be found in K. E. Iverson, *A Programming Language* (Wiley, 1962).

$$
\begin{aligned}
\sim x &\equiv 0 = x\\
x \vee y &\equiv x \lceil y\\
x \wedge y &\equiv x \lfloor y
\end{aligned}
$$

The identity elements of $\vee$ and $\wedge$ are 0 and 1 respectively.

The monadic functions $\lfloor$ and $\lceil$ are called *floor* and *ceiling* respectively, and are defined as follows:

$$
\begin{aligned}
\lfloor x &\equiv x - 1|x\\
\lceil x &\equiv - \lfloor - x
\end{aligned}
$$

In other words, $\lfloor x$ is the largest integer not exceeding x, and $\lceil x$ is the smallest integer not exceeded by x.

The dyadic function $\perp$ is called the *base value* function and is defined as follows: $\boldsymbol{r} \perp \boldsymbol{x} \equiv +/ \boldsymbol{w} \times \boldsymbol{x}$, where $\boldsymbol{w}_{\rho \boldsymbol{w}} \equiv 1$ and $\boldsymbol{w}_{i-1} \equiv \boldsymbol{r}_i \times \boldsymbol{w}_i$. For example, if $\boldsymbol{x} \equiv 2 , 1 , 15$ is a vector giving elapsed time in hours, minutes, and seconds, and if $\boldsymbol{r} \equiv 24 , 60 , 60$ is the corresponding *radix vector*, then $\boldsymbol{w} \equiv 3600 , 60 , 1$ and $\boldsymbol{r} \perp \boldsymbol{x} \equiv 7275$ is the elapsed time in seconds. If either argument is a scalar it is extended in the usual way. For example, $10 \perp \boldsymbol{d}$ is the base-10 value of the digits $\boldsymbol{d}$, and $\boldsymbol{r} \perp x$ is the polynomial in x with coefficients $⌽\, \boldsymbol{r}$ (see Exercise 4.36). The monadic function $\perp$ is defined as a special case of the dyadic function $\perp$:

$$\perp \boldsymbol{x} \equiv 2 \perp \boldsymbol{x}$$

In other words, $\perp \boldsymbol{x}$ is the base-2 value of $\boldsymbol{x}$.

The dyadic function $\top$ is the inverse of the base value function. Thus if $\boldsymbol{x} \leftarrow \boldsymbol{r} \top j$, then $\rho\, \boldsymbol{x} \equiv \rho\, \boldsymbol{r}$ and $\boldsymbol{r} \perp \boldsymbol{x} \equiv (\times / \boldsymbol{r})|j$. For example, $(24 , 60 , 60) \top 7275 \equiv 2 , 1 , 15$ and $(4\, \rho\, 10) \top 1776 \equiv 1 , 7 , 7 , 6$.

The *characteristic* function ϵ is defined as follows:

$$y \,\epsilon\, \boldsymbol{x} \equiv \vee / y = \boldsymbol{x}$$

In other words, $y \,\epsilon\, \boldsymbol{x}$ is equal to 1 if y is equal to some component of $\boldsymbol{x}$. More generally, if $\boldsymbol{y}$ is a vector the statement

$$\boldsymbol{u} \leftarrow \boldsymbol{y} \,\epsilon\, \boldsymbol{x}$$

specifies a vector $\boldsymbol{u}$ such that $\rho\, \boldsymbol{u} \equiv \rho\, \boldsymbol{y}$ and $\boldsymbol{u}_i \equiv \boldsymbol{y}_i \,\epsilon\, \boldsymbol{x}$. Hence the set $\boldsymbol{y}$ is included in $\boldsymbol{x}$ if and only if $\wedge / \boldsymbol{y} \,\epsilon\, \boldsymbol{x} \equiv 1$. In particular, $(\iota\, n) \,\epsilon\, \boldsymbol{x}$ denotes a logical vector of dimension n having 1's in the positions $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and so on. For example, $(\iota\, 8) \,\epsilon\, 2 , 3 , 5 \equiv 0 , 1 , 1 , 0 , 1 , 0 , 0 , 0$.

The dyadic function ι is called *inverse indexing*. The statement

$$\boldsymbol{j} \leftarrow \boldsymbol{a} \,\iota\, \boldsymbol{x}$$

is valid only if the set $\boldsymbol{a}$ contains the set $\boldsymbol{x}$ (that is, only if $\wedge / \boldsymbol{x} \in \boldsymbol{a} \equiv 1$); it specifies a vector $\boldsymbol{j}$ such that $\boldsymbol{x} \equiv \boldsymbol{a}_{\boldsymbol{j}}$.

If F and G are dyadic functions, then

$$\boldsymbol{x} F \,.\, G \,\boldsymbol{y} \equiv F / \boldsymbol{x} G \boldsymbol{y}$$

More generally, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are matrices such that $(\rho \boldsymbol{X})_2 \equiv (\rho \boldsymbol{Y})_1$, then the statement

$$\boldsymbol{Z} \leftarrow \boldsymbol{X} F \,.\, G \boldsymbol{Y}$$

specifies a matrix $\boldsymbol{Z}$ such that $\rho \boldsymbol{Z} \equiv (\rho \boldsymbol{X})_1, \; (\rho \boldsymbol{Y})_2$, and $\boldsymbol{Z}^i_j \equiv \boldsymbol{X}^i F \,.\, G \boldsymbol{Y}_j$. The matrix $\boldsymbol{Z}$ is called the *matrix product* of $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *with respect to F and G.* If F and G are *addition* and *multiplication* respectively, then $\boldsymbol{Z}$ is called the *ordinary* matrix product of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Either or both of the arguments of the function $F \,.\, G$ may be vectors.

The *outer product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $\boldsymbol{x} \circ . F \boldsymbol{y}$, where F is any dyadic function. The outer product is a matrix $\boldsymbol{M}$ of dimension $(\rho \boldsymbol{x}), \rho \boldsymbol{y}$ and $\boldsymbol{M}^i_j \equiv \boldsymbol{x}_i F \boldsymbol{y}_j$. For example, $(\iota 2) \circ . \times \iota 3$ is the matrix

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{matrix}$$

and $(\iota 3) \circ . = \iota 3$ is the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Exercises

9.1 Type each of the following four sequences on the console typewriter and observe and interpret the results produced.

```
X←3
Y←X×X
⎕←Y
```

```
X←⍳3
Y←X×X
⎕←Y
Z←Y÷Y[3]
⎕←Z
```

```
N←8
Z←⍳N
⎕←+/Z
⎕←×/Z
```

```
Z←⍳10
X←⌽Z
Y←X×Z
M←⌈/Y
⎕←M
⎕←M=Y
```

9.2 (a) Type the definition of the function U occurring in Figure 9.3.

(b) Execute the function U for various integral values of the arguments (for example, $\square \leftarrow 21\ U\ 28$), and verify that the result is the greatest common divisor of the arguments.

(c) Type the definitions of all the functions of Figure 9.3, and execute function A for various integral values of the arguments.

9.3 (a) Display each of the functions entered in Exercise 9.2.

(b) Use the method of correcting functions to change the first statement of the function A to

$$P \leftarrow X, Y, (X\ L\ Y), X\ U\ Y$$

(c) Display the modified function A.

(d) Execute the modified function A for the arguments 21 and 28.

(e) Execute the modified function A for several pairs of integral values of the arguments.

9.4 (a) Insert the statement

$$P \leftarrow P, (X, Y) \div P[4]$$

in the function A produced in Exercise 9.3 so that it becomes the second statement of the function (with the original two statements becoming statements 1 and 3 respectively).

(b) Display the modified function A.

(c) Execute the modified function A for various integral values of the arguments.

(d) $X \div Y$ is clearly equal to $P[3] \div P[4]$. What is the relation between the pairs (X, Y) and $P[3], P[4]$?

9.5 (a) Delete the second statement of the function A of Exercise 9.4.

(b) Display the resulting function A.

(c) Execute the resulting function for various integral values of the arguments.

9.6 (a) Type the definition of the function $a \leftarrow P\, k$ of Program 2.8 (b). (Note that $\sqrt{\ } x$ is replaced by $x * .5$.)

(b) Execute the statement $\square \leftarrow P\ K$ for various values of K, and compare the results with the known value of π. (If the results are not correct, insert statements of the form $\square \leftarrow X$ in the program to observe successive values of any chosen variable X in order to analyze and correct the behavior of the program.)

(c) Use the attention button to interrupt the execution of the function P, and then execute statements of the form $\square \leftarrow X$ to type out the values of each of the variables in the program.

(d) Use a branch statement to resume execution of the function P so that the final result will be correct.

9.7 (a) Modify the function P of Exercise 9.6 so as to use statement labels in each of the branches. Display and execute the modified program to ensure that it is correct.

(b) Use statements of the form $\square \leftarrow L$ to type out the value of each label variable L.

(c) Insert innocuous statements of the form $X \leftarrow X$ in the program so as to change the statement number associated with some label, execute the program to ensure that it still performs correctly, and then type out the new value of the label.

9.8 Use the heading

$$\nabla Q \ N$$

to define a function Q with the argument N which types out the line

THE PRIMES UP TO

followed on the next line by the value of N, and on the next line by the vector of the primes up to N. (Use the program of Exercise 2.28 (a).)

9.9 Enter the program of Exercise 2.30 (a) and use tracing to analyze its execution for a few values of the argument X.

9.10 Execute, analyze, and correct any errors in the following programs:

(a) Program 2.13

(b) Program 2.14

(c) The program for Exercise 2.12

(d) The program for Exercise 2.15

9.11 (a) A set of eight equal weights is known to be perfect except that one of them is too light. Describe in words a procedure for determining which of the weights is light, using not more than two weighings with a balance scale.

(b) Write and execute a program for the procedure of part (a). Use a vector W of dimension 8 to represent the given weights, and use an expression of the form

```
D← (+/W[1,2,3]) - +/W[6,7,8]
```

to determine the difference D in balancing the first three weights against the last three.

9.12 (a) A set of 12 equal weights is known to be perfect except that at most one of them may be either too light or too heavy. Describe in words a procedure for determining which weight (if any) is faulty and whether it is light or heavy, using at most three weighings on a balance scale.

(b) Write and execute a program for the procedure of part (a).

Appendix A

Conventions Governing Order of Evaluation

The common conventions for the evaluation of unparenthesized expressions include the rules that (1) in a multilevel expression such as $\frac{a+b}{c \div d}$, each line is evaluated before the function connecting the lines is evaluated; (2) subject to the first rule, multiplication and division are performed before addition and subtraction; (3) subject to the first two rules, evaluation proceeds from left to right; (4) division can be represented by three distinct but synonymous symbols $\left(a \div b,\ a\,/\,b,\ \text{and}\ \frac{a}{b}\right)$; and (5) multiplication can be represented by two distinct but synonymous symbols $(a \times b$ and $a \cdot b)$, or the symbol can be elided. The one convention used in this book is that (subject to parentheses) evaluation proceeds from right to left. This appendix treats the major reasons for this choice.

The common conventions are usually defended on the grounds that they are simple and well known and that their use significantly simplifies the reading and writing of expressions. Because of the familiarity of certain common constructions, these conventions appear simple, but this simplicity is illusory and vanishes on closer examination. Inquiries among students and colleagues have shown such disagreement on the interpretation of the conventions as to dispel the notion that they are well known. Finally, the much simpler convention adopted in this text proves at least as effective in simplifying the reading and writing of expressions.

Consider, for example, the expressions $x \div y \times z$ and $x \div yz$. According to the rules, both are equivalent to the expression $(x \div y) \times z$. However, yz is frequently used as an expression for multiplication which is performed first regardless of other rules. Furthermore, the dot notation for multiplication yields the expression $x \div y \cdot z$, which (according to the interpretations encountered) seems to fall midway between the other cases. Proponents of the common convention protest that such expressions would be parenthesized anyway for clarity; but then the convention seems to lose most of its value.

Matters are further complicated by the alternative notations for division. For example, $x \div y \div z$ and $x \div y\,/\,z$ should have the same interpretation, but frequently they do not. Similarly, the formally equivalent expressions $x+a\div y+b$ and $x+a/y+b$ frequently receive different interpretations. It is interesting to consider the different possible evaluations of the following expressions which, according to rule 3, are equivalent:

$$\begin{array}{ccc} x\div y\times z & x\div y\cdot z & x\div yz \\ x/y\times z & x/y\cdot z & x/yz \end{array}$$

The common convention also appears to include a number of tacit rules that writers obey automatically. For example, xy may be written for $x\times y$, and any variable should be replaceable by a numerical value. However, while the expression $3y$ is commonplace, most readers would find the expressions $x3$ and $3\ 4$ jarring and perhaps inadmissible as expressions for $x\times 3$ and 3×4.

In spite of these defects, the common conventions are reasonably convenient when applied to simple expressions involving only the four basic arithmetic functions, but more serious difficulties arise in their haphazard extension to other functions. For example, the expression $\sin n\times \cos m$ would be interpreted as $(\sin n)\times(\cos m)$, whereas $\sin n\times\pi$ would be interpreted as $\sin(n\times\pi)$. Moreover, the expression $a^{b^{c^{d}}}$ is usually interpreted as $a^{(b^{(c^{d})})}$ rather than as $((a^{b})^{c})^{d}$ (that is, from right to left rather than from left to right according to rule 3), apparently because the latter case can be expressed by the equivalent expression $a^{b\times c\times d}$. In the notation used in this book the first case would be expressed as either $a\ast b\ast c\ast d$ or $\ast/\,a\,,b\,,c\,,d$ and the second as either $a\ast b\times c\times d$ or $a\ast\times/\,b\,,c\,,d$.

As further functions are introduced (for example, absolute value, maximum, minimum, residue, the relations, logical functions, and the circular functions), the complexity grows and the utility of any relative priority of execution among the functions decreases. Mathematical texts handle this problem either by liberal use of parentheses or by *ad hoc* (and frequently unstated) conventions. Programming languages, which must face the issue more formally, have usually treated the problem by establishing a hierarchy of priorities among the functions such that any function is evaluated before all others having lower priorities. Such a system is usually very complex (Algol, one of the best known, has nine priority levels) and can therefore be used efficiently only by a programmer who employs it frequently. The occasional (and the prudent) programmer avoids the whole issue by including all the parentheses that would have been required with no convention.

Further examples of the complexity and ambiguity of the common conventions could be easily adduced. However, the skeptical reader will find it more instructive to scan various textbooks trying to formulate precisely the rules used (stated or implied) and applying them rigorously.

The question of the efficacy of the common convention in reducing the need for parentheses will now be addressed. Any convention will reduce the need for parentheses, but the important question is how the common convention compares in this respect with other conventions, and in particular with the notation used in this text.

The utility of the common convention stands forth well in the expression for a polynomial. For example, in the expression

$$ax^p + bx^q + cx^r$$

it would be awkward to have to enclose each term in parentheses. However, in the present notation this would be written as

$$+/\ (a\ ,b\ ,c) \times x \ast p\ ,q\ ,r$$

or, if the vectors of coefficients and exponents were denoted by $\boldsymbol{c}$ and $\boldsymbol{e}$ respectively, then it would be written as

$$+/\ \boldsymbol{c} \times x \ast \boldsymbol{e}$$

These forms make clear the structure of the polynomial while permitting suppression of detail by using vectors; the corresponding expression in conventional notation is

$$\boldsymbol{c}_1 \times x^{e_1} + \boldsymbol{c}_2 \times x^{e_2} + \ldots + \boldsymbol{c}_n \times x^{e_n},$$

where n is the magic variable that denotes the dimensions of all vectors.

The expression (derived in Chapter 4) for the efficient evaluation of a polynomial such as $(a\ ,b\ ,c\ ,d\ ,e\ ,f)\ \Pi\ x$ provides a further example. In the notation used in this text it appears (without parentheses) as

$$(a\ ,b\ ,c\ ,d\ ,e\ ,f)\ \Pi\ x \equiv a + x \times b + x \times c + x \times d + x \times e + x \times f$$

whereas in the common convention it would appear as

$$\begin{aligned}&(a\ ,b\ ,c\ ,d\ ,e\ ,f)\ \Pi\ x\\ &\qquad \equiv a + x \times\ (b + x \times\ (c + x \times\ (d + x \times\ (e + x \times f))))\end{aligned}$$

Further examples could be adduced, but again the skeptical reader will find it more instructive to formulate a set of *precise* rules based on the common convention and to translate into the resulting notation the expressions appearing in the present text.

There is one further argument against imposing a priority among functions in the present notation. If F and G are dyadic functions, then the expression $F/\,\boldsymbol{x}\,G\,\boldsymbol{y}$ would have either of two interpretations (that is, $(F/\,\boldsymbol{x})\,G\,\boldsymbol{y}$ or $F/\,(\boldsymbol{x}\,G\,\boldsymbol{y})$), depending upon the relative priorities of F and G. These two interpretations differ markedly in form and would therefore lead to confusion. For example, $+/\,\boldsymbol{x}\times\boldsymbol{y}$ would be interpreted as $+/\,(\boldsymbol{x}\times\boldsymbol{y})$ whereas the similar expression $\times/\,\boldsymbol{x}+\boldsymbol{y}$ would be interpreted as $(\times/\,\boldsymbol{x})+\boldsymbol{y}$. Similar remarks apply to the matrix product $\boldsymbol{M}\,F\,.\,G\,\boldsymbol{N}$ (defined in Chapter 9).

The reasons for choosing a right-to-left instead of a left-to-right convention are:

1. The usual mathematical convention of placing a monadic function to the left of its argument leads to a right-to-left execution for monadic functions; for example, $F\,G\,x \equiv F\,(G\,x)$.
2. The notation $F/\,\boldsymbol{z}$ for reduction (by any dyadic function F) tends to require fewer parentheses with a right-to-left convention. For example, expressions such as $+/\,(\boldsymbol{x}\times\boldsymbol{y})$ or $+/\,(\boldsymbol{u}/\boldsymbol{x})$ tend to occur more frequently than $(+/\,\boldsymbol{x})\times\boldsymbol{y}$ and $(+/\,\boldsymbol{u})\,/\,\boldsymbol{x}$.
3. An expression *evaluated* from right to left is the easiest to *read* from left to right. For example, the expression

   $$a+\underbrace{x\times b}+\underbrace{x\times c}+\underbrace{x\times d}+\underbrace{x\times e}+\underbrace{x\times f}$$

   (for the efficient evaluation of a polynomial) is read as a plus the entire expression following, or as a plus x times the following expression, or as a plus x times b plus the following expression, and so on.
4. In the definition

   $$F/\,\boldsymbol{x} \equiv \boldsymbol{x}_1\,F\,\boldsymbol{x}_2\,F\,\boldsymbol{x}_3\,F\ldots F\,\boldsymbol{x}_{\rho\,\boldsymbol{x}}$$

   the right-to-left convention leads to a more useful definition for nonassociative functions F than does the left-to-right convention. For example, $-/\,\boldsymbol{x}$ denotes the alternating sum of the components of $\boldsymbol{x}$, whereas in a left-to-right convention it would denote the first component minus the sum of the remaining components. Thus if $\boldsymbol{d}$ is the vector of decimal digits representing the number n, then the value of the expression $0=9|+/\,\boldsymbol{d}$ determines the divisibility of n by 9; in the right-to-left convention, the similar expression $0=11|-/\,\boldsymbol{d}$ determines divisibility by 11.

Tables of Circular Functions
0°–45°

deg	rad	sin	cos	tan	cot	rad	deg
0°	0.0000	0.0000	1.0000	0.0000	———	1.5708	90°
1°	0.0175	0.0175	0.9998	0.0175	57.2900	1.5533	89°
2°	0.0349	0.0349	0.9994	0.0349	28.6363	1.5359	88°
3°	0.0524	0.0523	0.9986	0.0524	19.0811	1.5184	87°
4°	0.0698	0.0698	0.9976	0.0699	14.3007	1.5010	86°
5°	0.0873	0.0872	0.9962	0.0875	11.4301	1.4835	85°
6°	0.1047	0.1045	0.9945	0.1051	9.5144	1.4661	84°
7°	0.1222	0.1219	0.9925	0.1228	8.1443	1.4486	83°
8°	0.1396	0.1392	0.9903	0.1405	7.1154	1.4312	82°
9°	0.1571	0.1564	0.9877	0.1584	6.3138	1.4137	81°
10°	0.1745	0.1736	0.9848	0.1763	5.6713	1.3963	80°
11°	0.1920	0.1908	0.9816	0.1944	5.1446	1.3788	79°
12°	0.2094	0.2079	0.9781	0.2126	4.7046	1.3614	78°
13°	0.2269	0.2250	0.9744	0.2309	4.3315	1.3439	77°
14°	0.2443	0.2419	0.9703	0.2493	4.0108	1.3265	76°
15°	0.2618	0.2588	0.9659	0.2679	3.7321	1.3090	75°
16°	0.2793	0.2756	0.9613	0.2867	3.4874	1.2915	74°
17°	0.2967	0.2924	0.9563	0.3057	3.2709	1.2741	73°
18°	0.3142	0.3090	0.9511	0.3249	3.0777	1.2566	72°
19°	0.3316	0.3256	0.9455	0.3443	2.9042	1.2392	71°
20°	0.3491	0.3420	0.9397	0.3640	2.7475	1.2217	70°
21°	0.3665	0.3584	0.9336	0.3839	2.6051	1.2043	69°
22°	0.3840	0.3746	0.9272	0.4040	2.4751	1.1868	68°
23°	0.4014	0.3907	0.9205	0.4245	2.3559	1.1694	67°
24°	0.4189	0.4067	0.9135	0.4452	2.2460	1.1519	66°
25°	0.4363	0.4226	0.9063	0.4663	2.1445	1.1345	65°
26°	0.4538	0.4384	0.8988	0.4877	2.0503	1.1170	64°
27°	0.4712	0.4540	0.8910	0.5095	1.9626	1.0996	63°
28°	0.4887	0.4695	0.8829	0.5317	1.8807	1.0821	62°
29°	0.5061	0.4848	0.8746	0.5543	1.8040	1.0647	61°
30°	0.5236	0.5000	0.8660	0.5774	1.7321	1.0472	60°
31°	0.5411	0.5150	0.8572	0.6009	1.6643	1.0297	59°
32°	0.5585	0.5299	0.8480	0.6249	1.6003	1.0123	58°
33°	0.5760	0.5446	0.8387	0.6494	1.5399	0.9948	57°
34°	0.5934	0.5592	0.8290	0.6745	1.4826	0.9774	56°
35°	0.6109	0.5736	0.8192	0.7002	1.4281	0.9599	55°
36°	0.6283	0.5878	0.8090	0.7265	1.3764	0.9425	54°
37°	0.6458	0.6018	0.7986	0.7536	1.3270	0.9250	53°
38°	0.6632	0.6157	0.7880	0.7813	1.2799	0.9076	52°
39°	0.6807	0.6293	0.7771	0.8098	1.2349	0.8901	51°
40°	0.6981	0.6428	0.7660	0.8391	1.1918	0.8727	50°
41°	0.7156	0.6561	0.7547	0.8693	1.1504	0.8552	49°
42°	0.7330	0.6691	0.7431	0.9004	1.1106	0.8378	48°
43°	0.7505	0.6820	0.7314	0.9325	1.0724	0.8203	47°
44°	0.7679	0.6947	0.7193	0.9657	1.0355	0.8029	46°
45°	0.7854	0.7071	0.7071	1.0000	1.0000	0.7854	45°
deg	rad	cos	sin	cot	tan	rad	deg

45°–90°

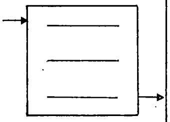
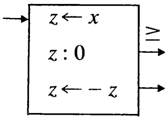
Tables of Base-10 Logarithms
1.00 – 5.49

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

Tables of Base-10 Logarithms
5.50–9.99

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Summary of Notation

Function	Notation	Definition or Example	Page Refs.	Computer Notation
Specification	$z \leftarrow x$	$z \leftarrow 3$ assigns the value 3 to z	6	$Z \leftarrow X$
Arithmetic	$+ - \times \div$		6	$+ - \times \div$
Branch	$\vert x : y \vert \xrightarrow{\mathscr{R}}$	Arrow is followed if $x \mathscr{R} y$ is true	11	$\rightarrow (X \mathscr{R} Y)/ S$
Relations $\mathscr{R}$	$< \leq = \geq > \neq$	in branches and relational functions	6	$< \leq = \geq > \neq$
Component of $\boldsymbol{x}$	$\boldsymbol{x}_i$	ith component of $\boldsymbol{x}$	18	$X\ [I]$
Dimension of $\boldsymbol{x}$	$\rho\ \boldsymbol{x}$	$\rho\ (3, 4, 5, 6) \equiv 4$	18	$\rho\ X$
Catenation	$\boldsymbol{x}, \boldsymbol{y}$	$\boldsymbol{x}, \boldsymbol{y} \equiv \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{\rho \boldsymbol{x}}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{\rho \boldsymbol{y}}$	18	X, Y
Definition of function F	$z \leftarrow F\ x$ 	$z \leftarrow \vert\ x$ 	39	$\nabla Z \leftarrow F\ X$ [1] ——— [2] ——— [3] ——— [4] ∇
Maximum	$x \lceil y$	$4 \lceil 2 \equiv 4$	40	$X \lceil Y$
Minimum	$x \lfloor y$	$4 \lfloor 2 \equiv 2$	43	$X \lfloor Y$
Residue	$m \vert n$	$3 \vert 7 \equiv 1;\ 3 \vert {}^{-}7 \equiv 2;\ 3 \vert 6 \equiv 0$	43	$X \vert Y$
Absolute value	$\vert\ x$	$\vert\ 3.14 \equiv 3.14;\ \vert {}^{-}3.14 \equiv 3.14$	43	$\vert X$
Negation	$- x$	$- x \equiv 0 - x$	43	$- X$
Exponentiation	$x \ast n$	$x \ast 0 \equiv 1;\ x \ast n \equiv x \times x \ast n - 1$	45	$X \ast N$
Factorial	$!\ n$	$!\ 0 \equiv 1;\ !\ n \equiv n \times !\ n - 1$	45	$!\ N$
Relation	$x \mathscr{R} y$	$(3 \leq 3) \equiv 1;\ (3 < 3) \equiv 0$	47	$X \mathscr{R} Y$
Compression	$\boldsymbol{u} / \boldsymbol{x}$	$(1, 0, 1, 0, 1)/ \boldsymbol{x} \equiv (\boldsymbol{x}_1, \boldsymbol{x}_3, \boldsymbol{x}_5)$	48	U / X
Reversal	⌽ $\boldsymbol{x}$	⌽ $1, 2, 3, 4 \equiv 4, 3, 2, 1$	48	⌽ X
Integer vector	$\iota\ n$	$\iota\ 4 \equiv 1, 2, 3, 4$	48	$\iota\ N$
Reduction	$F / \boldsymbol{x}$	$F / \boldsymbol{x} \equiv \boldsymbol{x}_1\ F\ \boldsymbol{x}_2\ F\ \boldsymbol{x}_3 \ldots F\ \boldsymbol{x}_{\rho \boldsymbol{x}}$	22	F / X
Row i of matrix	$\boldsymbol{M}^i$	$\boldsymbol{M}^2 \equiv 4, 5, 6$	76	$M\ [I\ ;]$
Column i of matrix	$\boldsymbol{M}_i$	$\boldsymbol{M}_2 \equiv 2, 5, 8, 11$	76	$M\ [;\ I]$
Element of matrix	$\boldsymbol{M}_j^i$	$\boldsymbol{M}_3^2 \equiv 6$ $\quad \left. \right\} \boldsymbol{M} \equiv \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{matrix}$	76	$M\ [I\ ; J]$
Restructuring	$\boldsymbol{d}\ \rho\ \boldsymbol{x}$	$(4, 3)\ \rho\ \iota\ 12 \equiv \boldsymbol{M}$ $12\ \rho\ \boldsymbol{M} \equiv \iota\ 12$	79	$D\ \rho\ X$
Polynomial	$\boldsymbol{c}\ \Pi\ x$	$\boldsymbol{c}_1 + (\boldsymbol{c}_2 \times x) + (\boldsymbol{c}_3 \times x^2) + \ldots$	62	
Natural exponential	$\ast\ x$	$\left(1, 1, \frac{1}{!2}, \frac{1}{!3}, \frac{1}{!4}, \ldots\right) \Pi\ x$	187	$\ast\ X$
Hyperbolic cosine	$A\ x$	$\left(1, 0, \frac{1}{!2}, 0, \frac{1}{!4}, \ldots\right) \Pi\ x$	180	
Hyperbolic sine	$B\ x$	$\left(0, 1, 0, \frac{1}{!3}, 0, \ldots\right) \Pi x$	180	
Cosine	$C\ x$	$\left(1, 0, \frac{{}^{-}1}{!2}, 0, \frac{1}{!4}, \ldots\right) \Pi\ x$	133	
Sine	$S\ x$	$\left(0, 1, 0, \frac{{}^{-}1}{!3}, 0, \frac{1}{!5}, \ldots\right) \Pi\ x$	133	
Tangent	$T\ x$	$(S\ x) \div C\ x$	145	
Hyperbolic tangent	$U\ x$	$(B\ x) \div A\ x$	180	
Base of natural logarithm	e	$2.71828 \ldots \equiv 1 + \frac{1}{!2} + \frac{1}{!3} + \ldots$	187	
Circular constant	π	$3.14159 \ldots$	15	

Index

DATE DUE

3/22/68 ~~APR 9~~ 1970			
JUN 8 1970			
AUG 20 1970			
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Function	Notation	Definition or Example	Page Refs.	Computer Notation
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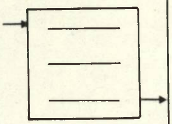
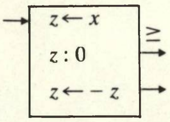
Catenation	$\boldsymbol{x}, \boldsymbol{y}$	$\boldsymbol{x}, \boldsymbol{y} \equiv \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{\rho \boldsymbol{x}}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{\rho \boldsymbol{y}}$	18	X, Y
Definition of function F	$z \leftarrow F\, x$ 	$z \leftarrow \lvert x$ 	39	$\nabla Z \leftarrow F\ X$ [1] ——— [2] ——— [3] ——— [4] ∇
Maximum	$x \lceil y$	$4 \lceil 2 \equiv 4$	40	$X \lceil Y$
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Residue	$m \vert n$	$3 \vert 7 \equiv 1;\ 3 \vert {}^-7 \equiv 2;\ 3 \vert 6 \equiv 0$	43	$X \vert Y$
Absolute value	$\vert x$	$\vert 3.14 \equiv 3.14;\ \vert {}^-3.14 \equiv 3.14$	43	$\vert X$
Negation	$- x$	$- x \equiv 0 - x$	43	$- X$
Exponentiation	$x \ast n$	$x \ast 0 \equiv 1;\ x \ast n \equiv x \times x \ast n - 1$	45	$X \ast N$
Factorial	$!\, n$	$!\,0 \equiv 1;\ !\,n \equiv n \times !\,n - 1$	45	$!\,N$
Relation	$x \mathscr{R} y$	$(3 \leq 3) \equiv 1;\ (3 < 3) \equiv 0$	47	$X \mathscr{R} Y$
Compression	$\boldsymbol{u} / \boldsymbol{x}$	$(1, 0, 1, 0, 1)/ \boldsymbol{x} \equiv (\boldsymbol{x}_1, \boldsymbol{x}_3, \boldsymbol{x}_5)$	48	U / X
Reversal	⌽ $\boldsymbol{x}$	⌽ $1, 2, 3, 4 \equiv 4, 3, 2, 1$	48	⌽ X
Integer vector	$\iota\, n$	$\iota\, 4 \equiv 1, 2, 3, 4$	48	$\iota\, N$
Reduction	$F/ \boldsymbol{x}$	$F/ \boldsymbol{x} \equiv \boldsymbol{x}_1\, F\, \boldsymbol{x}_2\, F\, \boldsymbol{x}_3 \ldots F\, \boldsymbol{x}_{\rho \boldsymbol{x}}$	22	F/ X
Row i of matrix	$\boldsymbol{M}^i$	$\boldsymbol{M}^2 \equiv 4, 5, 6$ $\boldsymbol{M} \equiv \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{matrix}$	76	$M\,[I\,;]$
Column i of matrix	$\boldsymbol{M}_i$	$\boldsymbol{M}_2 \equiv 2, 5, 8, 11$	76	$M\,[;\,I]$
Element of matrix	$\boldsymbol{M}^i_j$	$\boldsymbol{M}^2_3 \equiv 6$	76	$M\,[I\,; J]$
Restructuring	$\boldsymbol{d}\, \rho\, \boldsymbol{x}$	$(4, 3)\, \rho\, \iota\, 12 \equiv \boldsymbol{M}$ $12\, \rho\, \boldsymbol{M} \equiv \iota\, 12$	79	$D\, \rho\, X$
Polynomial	$\boldsymbol{c}\, \Pi\, x$	$\boldsymbol{c}_1 + (\boldsymbol{c}_2 \times x) + (\boldsymbol{c}_3 \times x^2) + \ldots$	62	
Natural exponential	$\ast\, x$	$\left(1, 1, \frac{1}{!2}, \frac{1}{!3}, \frac{1}{!4}, \ldots\right) \Pi\, x$	187	$\ast\, X$
Hyperbolic cosine	$A\, x$	$\left(1, 0, \frac{1}{!2}, 0, \frac{1}{!4}, \ldots\right) \Pi\, x$	180	
Hyperbolic sine	$B\, x$	$\left(0, 1, 0, \frac{1}{!3}, 0, \ldots\right) \Pi\, x$	180	
Cosine	$C\, x$	$\left(1, 0, \frac{^-1}{!2}, 0, \frac{1}{!4}, \ldots\right) \Pi\, x$	133	
Sine	$S\, x$	$\left(0, 1, 0, \frac{^-1}{!3}, 0, \frac{1}{!5}, \ldots\right) \Pi\, x$	133	
Tangent	$T\, x$	$(S\, x) \div C\, x$	145	
Hyperbolic tangent	$U\, x$	$(B\, x) \div A\, x$	180	
Base of natural logarithm	e	$2.71828 \ldots \equiv 1 + \frac{1}{!2} + \frac{1}{!3} + \ldots$	187	
Circular constant	π	$3.14159 \ldots$	15	